# STATISTICALLY CONSISTENT SYNTHETIC COVARIANCE TO QUANTIFY OPERATIONAL UNCERTAINTY

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## ABSTRACT

For a number of reasons, quality covariance information is often unavailable to space operators when assessing collision probabilities. A proposed method to address these issues is the concept of synthetic covariances. This involves computing a covariance from an existing set of position and velocity data. As an alternative to traditional approaches, the method treats the state vector in an analogous manner to an observation. The key to the approach is the filtering of ephemeris with a priori noise information and the adjustment of the residuals for consistency. This work reviews some of the possible approximations to this problem to enable the generation of a state estimate. The state estimate and its covariance are derived solely from the time series state vector source, in this case Two Line Element sets. The performance of the algorithms will be evaluated in terms of covariance containment for state prediction, mimicking conjunction analysis scenarios in routine space operations.

## 1. INTRODUCTION

Position and velocity covariances are critical for assessing collision probabilities in space operations; however, they are often unavailable or questionably estimated by flight dynamics systems. Spacecraft operators frequently encounter “black box” systems that do not readily output covariance matrices, or they may receive covariances tied to the Orbit Determination (OD) epoch lacking time history or customization options regarding reference frames and parameters. In addition, organizations providing Space Situational Awareness (SSA) information are hesitant to disclose positional accuracy metrics. As a result, reliable covariance information is scarce.

To address these limitations, we embrace the concept of synthetic covariances, computed directly from a given time series of position and velocity estimates. Traditional approaches for constructing synthetic covariances are mainly based on standard overlap comparisons. In this regard, there are methods based on predefined error functions and covariance matrix eigenvector reference frames. Recently, a novel digital statistics method for 6x6 covariance uncertainty modeling was presented drawing on statistics from time and argument-of-latitude bins on an object-by-object basis [1].

Alternatively, we propose to pose an estimation problem where the state vector is used in a way analogous to an observation. The solution approach consists of an initial filtering of the available ephemeris with or without a priori noise information, that is adaptively adjusted for the residuals to be statistically consistent. The estimation of noise covariance matrices in the optimal filtering problem has been the subject of research in the last 50 years [2], mainly in the context of KF and linear invariant systems. However, some of them have been extended to nonlinear problems. In general, the methods can be classified into different groups according to the approach [3] In this work, we will review some of the possible approximations [4, 5] applied to this specific problem. In this way, it is possible to generate a state estimate, consisting of the expected state and its uncertainty represented by the covariance, compatible with the dynamical model and the filtering method used, solely from the time series of state vectors.

We will evaluate the performance of this new technique in a similar way to previous work [1]. Two-Line Elements (TLEs) will be used both for estimation and uncertainty quantification analysis, as we will emphasize on the prediction capabilities of the proposed noise estimation approaches.

## 2. LITERATURE REVIEW

Covariance realism is a topic that has been a subject of research in the last years. In the white paper published in 2016, Poore et al. [6] established the boundaries and challenges of improving uncertainty knowledge in the orbit determination process.

From a comprehensive perspective, the subject can be framed in the domain of Uncertainty Quantification (UQ). The field is growing with application in many scientific and engineering problems [7]. The definition proposed by the National Research Council is the follow-

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ing: "The process of quantifying uncertainties associated with model calculations of true, physical quantities of interest, with the goals of accounting for all sources of uncertainty and quantifying the contributions of specific sources to the overall uncertainty" [8]. As such, it is a broad field transversal to all engineering subjects. However, UQ methods are highly dependent on the application domain, given that they are usually related to the physical processes involved. In orbital propagation, a fairly deterministic field, on the other hand, UQ is a novel topic. A seminal work is the PhD thesis of Lamberto Dell'Elce. In 2019, there was already a (consolidated) framework to address UQ in orbital mechanics [9]. Throughout the last years, there has been an increased interest in the topic, due to the impact of uncertainty in the provision of SST services from cataloging to conjunction assessment. Within space engineering, UQ is often referred to as Uncertainty Realism. Provided the state uncertainty is generally assumed Gaussian in operational contexts, UQ translates to Covariance Realism for practical applications. The majority of covariance realism methods rely on the fitting of some parameters that characterize the process noise [10]. However, process noise is not accounted for in the most commonly used method for orbit determination: batch least squares. Several workarounds to this particular problem have been proposed, such as extending the formulation to explicitly account for the process noise [11], or using the noise of the so-called consider parameters as a proxy [12, 13].

In discrete-time sequential filters, such as the Kalman Filter, process noise accounts for the unmodeled dynamics, i.e., the difference between the acceleration model used for estimation and the actual acceleration experienced by the spacecraft. Note it is relevant to properly estimate the process noise, as it is a diffusive term that inflates the propagated uncertainty, thereby affecting how measurements and dynamics are combined to update the knowledge on the state of an object. Therefore, covariance realism can be regarded as the result of a proper estimation of the process noise. In sequential filters, process noise is usually modeled as an additive Gaussian noise with zero mean and covariance $Q$. The tuning of this matrix has been a subject of research from the early times of the Kalman Filter [14]. Adaptive filtering techniques allow us to estimate the process noise covariance. There are multiple approaches that can be used for process noise estimation, and they can be divided into four categories [3]: Bayesian [15, 16], maximum likelihood [17], correlation or autocovariance [18] and covariance matching [19, 20]. Nevertheless, not all the methods proposed in the literature are of applicability in the orbit determination problem. Some of the assumptions made on these approaches do not hold (in general) for SST, as is the case for linear time-invariant dynamics, feedback-free methods that do not update the process noise covariance or non-correlated dynamical noise. In response to the specific needs of SST, Stacey et al. [4] devised methods that meet the operational requirements of orbit determination.

Moreover, there could be applications where the observation of measurement noise is not well characterized a priori, either due to the sensor characteristics or the data source. The latter is in fact the case for the most thorough publicly available data source for SSA: Two Line Element sets (TLEs). A single TLE contains the expected state and (up to certain point) short term dynamical characteristics of a given space object, as estimated by the US 18th Space Defense Squadron (SDS). Methods capable of jointly estimating the process and measurement noise are thus deemed necessary for a fully autonomous processing. Among the different proposals in the tracking literature, the one by Dunik et al. [5] appears to be consistent and operationally viable. Therein, the authors propose to derive the process and measurement noise levels based on the statistics of the observation residuals, i.e. the Measurement Prediction Error (MPE). The algorithm, dubbed Noise covariance matrices Estimation with Gaussianity Assessment (NEGA) is expected to perform well in nearly linear environments where the process and measurement noise statistics remain approximately constant.

According to the above discussion, two methods have been selected for implementation: one suitable for sequential filters [4] and one developed for time-variant linear systems [3]. Their description can be found in the following sections.

## 3. NOISE ESTIMATION

### 3.1. Problem Statement

The evolution of the satellite state can be modeled as a stochastic dynamical system with an additive Wiener process. In Itô calculus,

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\mathrm{d}t + \sigma_w(\mathbf{x})\mathrm{d}W\,. \tag{1}$$

The value of $\sigma_w$ provides the size of the Wiener process and, it is related to the process noise and, in this work, it is *a priori* unknown.

A reference trajectory is selected, $\mathbf{x}_{ref}$, such that

$$\mathrm{d}\mathbf{x}_{ref} = \mathbf{f}(\mathbf{x}_{ref}, t)\mathrm{d}t\,. \tag{2}$$

The reference trajectory can be the output of a filter, i.e. the expected state at a certain time, or given *a priori*. The relative state is defined as: $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_{ref}$. Assuming

$$\mathbf{f}(\mathbf{x}_{ref} + \delta\mathbf{x}, t) \approx \mathbf{f}(\mathbf{x}_{ref}, t) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{ref}} \delta\mathbf{x}\,, \tag{3}$$

the relative dynamics equation can thus be expressed as

$$\mathrm{d}\delta\mathbf{x} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{ref}} \delta\mathbf{x}\mathrm{d}t + \sigma_w(\mathbf{x})\mathrm{d}W\,. \tag{4}$$

The integral between two instants of time, $t_k$ and $t_{k+1}$ reads

$$\delta\mathbf{x}(t_{k+1}) = \Phi(t_{k+1}, t_k)\delta\mathbf{x}(t_k) + \int_{t_k}^{t_{k+1}} \sigma_w(\mathbf{x})\mathrm{d}W\,, \tag{5}$$

where $\Phi(t_{k+1}, t_k) = \int_{t_k}^{t_{k+1}} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big|_{\mathbf{x}_{ref}} \mathrm{d}t$. Defining a Gaussian noise $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ and a state noise shaping matrix $\mathbf{G}_k$ as

$$\mathbf{G}_k \mathbf{w}_k = \int_{t_k}^{t_{k+1}} \sigma_w(\mathbf{x}) \mathrm{d}W \,, \tag{6}$$

the resulting discrete time dynamics model is

$$\delta\mathbf{x}(t_{k+1}) = \Phi_{k+1,k}\delta\mathbf{x}(t_k) + \mathbf{G}_k\mathbf{w}_k \,, \tag{7}$$

with $\Phi_{k+1,k} = \Phi(t_{k+1}, t_k)$. The continuous-time process noise has been alternatively modeled in other works [4] for the SNC approach as $\varepsilon$, a zero-mean white Gaussian process, whose autocovariance is:

$$E\left[\epsilon(t)\epsilon(t)^T\right] = \tilde{\mathbf{Q}}(t)\delta(t-\tau) \tag{8}$$

The corresponding process noise mapping matrix is $\mathbf{\Gamma}$ in [4]. With this notation, both noise models are equivalent with $\epsilon \sim \mathcal{N}(\mathbf{0}, \tilde{\mathbf{Q}})$, and $\mathbf{G}_k = \int_{t_k}^{t_{k+1}} \Phi(t_k, \tau)\mathbf{\Gamma}(\tau)\mathrm{d}\tau$.

The description of the model is completed with the measurement equation:

$$\mathbf{z} = \mathbf{h}(\mathbf{x}, t) + \nu \,. \tag{9}$$

with $\nu \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$. Note $\mathbf{R}$ is assumed unknown for the problem of interest. The reference trajectory generates, through the measurement model without noise, a reference measurement

$$\mathbf{z}_{ref} = \mathbf{h}(\mathbf{x}_{ref}, t) \,. \tag{10}$$

Approximating

$$\mathbf{h}(\mathbf{x}_{ref} + \delta\mathbf{x}, t) \approx \mathbf{h}(\mathbf{x}_{ref}, t) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{ref}} \delta\mathbf{x} \,, \tag{11}$$

the resulting linear measurement model reads

$$\delta\mathbf{z}(t_k) = \mathbf{z}(t_k) - \mathbf{z}_{ref}(t_k) = \mathbf{H}\delta\mathbf{x}(t_k) + \nu_k \,, \tag{12}$$

where $\mathbf{H} = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}_{ref}}$.

### 3.2. Process Noise Estimation. ASNC.

The method included in this section corresponds to the Adaptive State Noise Compensation (ASNC) described in [4]. Therein, two new possible approaches are proposed to tackle the problem of estimating the process noise, overcoming the disadvantages detected in the state-of-the-art in the field of orbit determination. The two methods are based on State Noise Compensation (SNC) and Dynamic Model Compensation (DMC), respectively. They combine the fundamentals of these approaches with a covariance matching method using a constrained weighted least squares optimization. Both methods are relatively similar in terms of applicability and performance so we only included ASNC for the sake of comparison. Note it is also expected to be more robust since it drops the estimation of a general (exponential) acceleration that could potentially lead to filter instabilities under certain circumstances. In what follows, a summary of ASNC is included for completeness, although it is possible to find all the details in Stacey and D'Amico [4].

Although ASNC is derived for linear systems, it can be applied to EKF or UKF, as these undergo a linearization of the dynamical and measurement models around the a priori estimate. The process noise covariance matrix

$$\mathbf{Q}_k = \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k, \tau)\mathbf{\Gamma}(\tau)\tilde{\mathbf{Q}}(\tau)\mathbf{\Gamma}(\tau)^T\mathbf{\Phi}(t_k, \tau)^T d\tau \tag{13}$$

enters into the time update step of the Kalman filter through the predicted covariance

$$\mathbf{P}_{k|k-1} = \mathbf{\Phi}_k\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_k^T + \mathbf{Q}_k \tag{14}$$

where $\mathbf{P}_{k|k-1}$ is the predicted covariance matrix at the current step conditioned on the measurement sequence through time step $k-1$, and $\mathbf{\Phi}_k = \mathbf{\Phi}(t_k, t_{k-1})$ is the state transition matrix from step $k-1$ to step $k$. The problem, thus, consists in estimating the process noise covariance matrix consistently. One option, based on a covariance matching approach, proposed by Myers and Tapley [19] approximates the process noise covariance matrix as:

$$\hat{\mathbf{Q}}_k = \frac{1}{N}\sum_{p=k-N}^{k-1}\left(\mathbf{P}_{p|p} - \mathbf{\Phi}_p\mathbf{P}_{p-1|p-1}\mathbf{\Phi}_p^T + \mathbf{\Delta}_p^x\mathbf{\Delta}_p^{x^T}\right) \,, \tag{15}$$

where

$$\mathbf{\Delta}_k^x = \mathbf{K}_k\mathbf{\Delta}_k^z, \; \mathbf{\Delta}_k^z = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_k \tag{16}$$

Therein,

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T\mathbf{S}_k^{-1} \tag{17}$$

is the so-called Kalman gain and

$$\mathbf{S}_k = E[\mathbf{\Delta}_k^z\mathbf{\Delta}_k^{zT}] = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k \tag{18}$$

is the innovation covariance. However, covariance matching presents several shortcomings, one of them is the lack of guarantee of the semi-positive definiteness of the process noise matrix. To overcome these drawbacks, Stacey and D'Amico [4] proposed the ASNC and ADMC methods, in which, at each time step, the methods find "the positive semi-definite matrix ($\tilde{\mathbf{Q}}$) that minimizes the difference in a weighted least squares sense between the elements of the spacecraft state process noise covariance" and the corresponding estimate obtained from Eq. (10). The optimization problem can be then solved using active set or interior point methods (c.f. [4]). As an alternative, and based on certain assumptions on the weighting matrix, the authors derive an efficient method to compute the update of the process noise covariance.

The ASNC algorithm can be summarized into the following four steps as a function of the window length $N$ used to derive the statistics of the Kalman filter state update.

- **Step 1** Solve for $\hat{\mathbf{Q}}_k$, using

$$\hat{\mathbf{Q}}_k = \frac{1}{N} \sum_{p=k-N}^{k-1} \left( \mathbf{\Delta}_p^x \mathbf{\Delta}_p^{x^T} \right) , \tag{19}$$

where $\mathbf{\Delta}_p^x = \mathbf{K}_p \mathbf{\Delta}_p^z$ is the Kalman state update. If the variables are not available from the EKF / UKF,

$$\mathbf{P}_{p|p-1} = \mathbf{\Phi}_p \mathbf{P}_{p-1|p-1} \mathbf{\Phi}_p^T + \hat{\mathbf{Q}}_k \tag{20}$$

$$\mathbf{S}_p = \mathbf{H}_p \mathbf{P}_{p|p-1} \mathbf{H}_p^T + \mathbf{R}_p \tag{21}$$

$$\mathbf{K}_p = \mathbf{P}_{p|p-1} \mathbf{H}_p^T \mathbf{S}_p^{-1} \tag{22}$$

- **Step 2** Find the weighting matrix for each component of the process noise matrix based on the considered measurement window

$$\mathbf{W} = \frac{1}{N^2} \sum_{p=k-N+1}^{k} \mathcal{W}_p \tag{23}$$

$$\mathcal{W}_p = \text{diag} \left( \text{vech}(\bar{\mathbf{\Sigma}}_p) \right) \tag{24}$$

$$\mathbf{\Sigma}_p = \mathbf{K}_p \mathbf{S}_p \mathbf{K}_p^T \tag{25}$$

$$\bar{\mathbf{\Sigma}}_p = \mathbf{\Sigma}_p^{o2} + \mathbf{\Sigma}_p^{diag} \mathbf{\Sigma}_p^{diag^T} \tag{26}$$

- **Step 3** Solve the quadratic programming problem, i.e. find the $\tilde{\mathbf{Q}}_k$ from SNC that matches the $\mathbf{Q}$ given by covariance matching assumptions

$$\tilde{\mathbf{Q}}_k : \underset{\tilde{\mathbf{Q}}}{\text{argmin}} (\mathbf{X}\tilde{\mathbf{Q}} - \mathbf{b})^T \mathbf{W}^{-1} (\mathbf{X}\tilde{\mathbf{Q}} - \mathbf{b}) \tag{27}$$

$$\mathbf{X}'_{i,j,l} = \int_{t_{k-1}}^{t_k} [\mathbf{\Phi}(t_k, \tau)\mathbf{\Gamma}(\tau)]_{i,l} [\mathbf{\Phi}(t_k, \tau)\mathbf{\Gamma}(\tau)]_{j,l} d\tau \tag{28}$$

for $i, j$ in $1, ..., n_{\mathbf{x}}$ and $l$ in $1, ..., dim(\tilde{\mathbf{Q}})$

$$\mathbf{X}_{:,l} = \text{vech} \left( \mathbf{X}'_{:,:,l} \right) \tag{29}$$

$$\mathbf{b} = \text{vech}(\hat{\mathbf{Q}}_{\mathbf{k+1}}), \tag{30}$$

- **Step 4** Compute the process noise at the subsequent step based on the expected PSD $\tilde{\mathbf{Q}}_k$

$$\mathbf{Q}_{k+1} = \text{vech}^{-1}(\mathbf{X}\tilde{\mathbf{Q}}_k) \tag{31}$$

Therefore, this filter can be run alongside an EKF or UKF and will provide adaptive process noise estimates once the filter has received enough observations, i.e for $k > N$. As described, note it assumes the measurement noise is known a priori so it would be left out to the user to tune these values offline in order to yield consistent state estimates.

### 3.3. Measurement and Process Noise Estimation. NEGA

In this method, a moving window of analysis of a fixed length (within a larger interval of analysis) is defined to produce Augmented Measurement Prediction Errors (AMPE). Then, from the statistical analysis of the AMPE, one can relate the unknown components of measurement and process noise and the covariance matrix of the AMPE in a linear fashion. A sample-based estimation of the AMPE covariance matrix allows us to estimate measurement and process noise. In what follows, the steps to relate the covariance matrix of the AMPE with the unknown noise covariance matrices and the known system elements are explained. Then, it is possible to describe how to estimate the noise covariances from the sample-based AMPE covariance. The summary included here can be expanded in the original works [3, 5].

The interval of analysis is $t \in [t_0, \tau]$. Each window of analysis $k$ is defined between time $t_k$ and $t_{k+L-1}$. The reference trajectory in the window of analysis, back-propagated from $\mathbf{x}_{ref}(t_{k+L-1})$, is computed from the estimated state in window $k-1$. That is,

$$\mathbf{x}_{ref}(t_{k+L-1}) = \mathbf{x}_{ref}(t_{k+L-2}) + \delta\hat{\mathbf{x}}_{k+L-2} + \int_{t_{k+L-2}}^{t_{k+L-1}} \mathbf{f}(\mathbf{x}, t) \mathrm{d}t. \tag{32}$$

The way in which the estimated state in window $k-1$ is computed is described below ( Eq. (47)).

The reference states in window $k$ are obtained as

$$\mathbf{x}_{ref}(t_{k+i}) = \int_{t_{k+L-1}}^{t_{k+i}} \mathbf{f}(\mathbf{x}, t) \mathrm{d}t + \mathbf{x}_{ref}(t_{k+L-1}) , \quad \forall i \in [0, ..., L-2]. \tag{33}$$

Therefore, given

$$\mathbf{X}_{k,ref}^L = [\mathbf{x}_{ref}^T(t_k), \mathbf{x}_{ref}^T(t_{k+1}), ..., \mathbf{x}_{ref}^T(t_{k+L-1})]^T, \tag{34}$$

the linear state-space model reads as Eqs. (7) and (12), with $\delta\mathbf{X}_k^L = \mathbf{X}_k^L - \mathbf{X}_{k,ref}^L$, and $\delta\mathbf{Z}_k^L = h(\mathbf{X}_k^L) - h(\mathbf{X}_{k,ref}^L)$.

With this notation, the procedure described in [3] can be followed with minor changes in the definition of the matrices involved. The augmented measurement prediction is written as

$$\delta\hat{\mathbf{Z}}_k^L = \mathcal{O}_k^L \Phi_{k+L-1,k+L-2} \left( \mathcal{O}_{k-1}^L \right)^\dagger \delta\mathbf{Z}_{k-1}^L . \tag{35}$$

The observability matrix in the window of analysis $k$ is defined as

$$\begin{aligned} \mathcal{O}_{k-1}^L = \big[ & (\mathbf{H}_k \Phi(t_k, t_{k+L-1}))^T , \\ & (\mathbf{H}_{k+1} \Phi(t_{k+1}, t_{k+L-1}))^T , ..., \\ & (\mathbf{H}_{k+L-1})^T \big]^T, \end{aligned} \tag{36}$$

with $\mathcal{O}_{k-1}^L \in \mathbb{R}^{Ln_z \times n_x}$. Then, the augmented measurement prediction error (AMPE) in window $k$ is defined as

$$\delta\tilde{\mathbf{Z}}_k^L = \delta\mathbf{Z}_k^L - \delta\hat{\mathbf{Z}}_k^L \tag{37}$$

Using the state-space model and the expression for the augmented measurement prediction, the AMPE can be computed as

$$\begin{aligned}\delta\tilde{\mathbf{Z}}_k^L =& \mathbf{\Gamma}_k^L\tilde{\mathbf{W}}_k^L + \tilde{\mathbf{V}}_k^L + \mathcal{O}_k^L\tilde{\mathbf{w}}_{k-1} \\ &- \mathcal{O}_k^L\Phi(t_{k+L-1}, t_{k+L-2})\left(\mathcal{O}_{k-1}^L\right)^\dagger \\ &\left(\mathbf{\Gamma}_{k-1}^L\tilde{\mathbf{W}}_{k-1}^L + \mathbf{V}_{k-1}^L\right) ,\end{aligned} \tag{38}$$

where the substitution $\tilde{\mathbf{w}}_k = \mathbf{G}_k\mathbf{w}_k \in \mathbb{R}^{n_x}$ was used and the vectors and matrices are defined by Eqs. (51-53).

With these definitions, the AMPE can be written in a compact form

$$\delta\tilde{\mathbf{Z}}_k^L = \mathcal{A}_k\xi_k^{L+} , \tag{39}$$

where $L+ = L+1$,

$$\xi_k^{L+} = \left[\left(\mathbf{W}_{k-1}^{L+}\right)^T, \left(\mathbf{V}_{k-1}^{L+}\right)^T\right]^T , \tag{40}$$

and $\mathcal{A}_k$ is defined in [3] Eqs. (14), (15) and (18).

Let $\mathbf{C}_k \in \mathbb{R}^{Ln_z \times Ln_z}$ be the covariance matrix of the AMPE $\tilde{\mathbf{Z}}_k^L$ defined by

$$\mathbf{C}_k = \mathrm{E}[\tilde{\mathbf{Z}}_k^L(\tilde{\mathbf{Z}}_k^L)^T] = \mathcal{A}_k\mathrm{E}[\xi_k^{L+}(\xi_k^{L+})^T]\mathcal{A}_k^T = \mathcal{A}_k\mathbf{\Xi}\mathcal{A}_k^T . \tag{41}$$

Following the algebraic manipulations described in [3], in which the matrix identity $\mathbf{ABC} = (\mathbf{C}^T \otimes \mathbf{A})\mathbf{B}_S$ ($\otimes$ is the Kronecker product and the notation $(\mathbf{A})_S$ stands for the columnwise stacking of a symmetric matrix $\mathbf{A} \in \mathbb{R}^{n_A \times n_A}$ into a vector $(\mathbf{A})_S \in \mathbb{R}^{n_A^2}$), the matrix $\mathbf{C}_k$ can be rewritten into

$$(\mathbf{C}_k)_S = (\mathcal{A}_k \otimes \mathcal{A}_k)\Xi_S = \mathbf{b}_k . \tag{42}$$

Therefore, in a compact form, the CM $(\mathbf{C}_k)_S$ is written as a linear function of the unknown noise covariance matrices $\mathbf{Q}, \mathbf{R}$ as

$$\mathbf{\Lambda}_k\theta = \mathbf{b}_k , \tag{43}$$

where $\mathbf{\Lambda}_k \in \mathbb{R}^{n_b \times n_\theta}$, with $n_b = (Ln_z)^2$, is the matrix depending on the known model matrices defined as

$$\mathbf{\Lambda}_k = (\mathcal{A}_k \otimes \mathcal{A}_k)\mathbf{\Psi} . \tag{44}$$

$\mathbf{b}_k \in \mathbb{R}^{n_b}$ and $\theta \in \mathbb{R}^{n_\theta}$, with $n_\theta = [n_w(n_w+1) + n_z(n_z+1)]/2$ is the vector of all unknown and unique elements of the noise covariance matrices defined as

$$\theta = [(\mathbf{Q}_{TS})^T, (\mathbf{R}_{TS})^T] . \tag{45}$$

The notation $\mathbf{A}_{TS}$ defined in [3] stands for the columnwise stacking of the unique $n_A(n_A+1)/2$ elements of the symmetric matrix $\mathbf{A} \in \mathbb{R}^{n_A \times n_A}$ by elimination of the supradiagonal elements. The matrix $\mathbf{\Psi}$ is knwon as the duplication or shape matrix, fulfilling the equation

$$\mathbf{\Xi}_S = \mathbf{\Psi}\theta . \tag{46}$$

Finally, the estimated state in window $k$ is computed from the measurements in that window

$$\delta\hat{\mathbf{x}}_{k+L-1} = \left(\mathcal{O}_k^L\right)^\dagger \delta\mathbf{Z}_k^L \tag{47}$$

After building the matrices for a window of analysis, the process is repeated for $k = 1, ..., \tau - L + 1$. The AMPE covariance matrix can thus be summarized for all windows of analysis as

$$\mathbf{\Lambda}\theta = \mathbf{b} \tag{48}$$

with $\mathbf{\Lambda} = [\mathbf{\Lambda}_1^T, \mathbf{\Lambda}_2^T, , ..., \mathbf{\Lambda}_{\tau-L+1}^T], \mathbf{b} = [\mathbf{b}_1^T, \mathbf{b}_2^T, , ..., \mathbf{b}_{\tau-L+1}^T]$.

Now, $\mathbf{C}_k$ can be estimated from a sequence of the measured and input data, and therefore the vector $\mathbf{b}$. Given a sequence of measurement predictor errors $[\tilde{\mathbf{Z}}_1^L, \tilde{\mathbf{Z}}_2^L, ..., \tilde{\mathbf{Z}}_{\tau-L+1}^L]$, the sample-based estimate of the vector $\mathbf{b}$ is given by

$$\hat{\mathbf{b}}_k = \tilde{\mathbf{Z}}_k^L \otimes \tilde{\mathbf{Z}}_k^L \tag{49}$$

Assuming that $\mathbf{\Lambda}$ is of full rank, the least squares optimum of the vector of the unknown elements of the noise covariance matrices is given by

$$\hat{\theta} = \Lambda^\dagger\hat{\mathbf{b}} . \tag{50}$$

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$$\tilde{\mathbf{W}}_k^L = \left[\tilde{\mathbf{w}}_k^T, \tilde{\mathbf{w}}_{k+1}^T, ..., \tilde{\mathbf{w}}_{k+L-1}^T\right]^T \tag{51}$$

$$\mathbf{V}_k^L = \left[\mathbf{v}_k^T, \mathbf{v}_{k+1}^T, ..., \mathbf{v}_{k+L-1}^T\right]^T \tag{52}$$

$$\mathbf{\Gamma}_k^L = \begin{bmatrix} \mathbf{0}_{n_z\times n_x} & \mathbf{H}_k & \cdots & \mathbf{H}_k\Phi(t_{k+1}, t_{k+L-2}) & \mathbf{H}_k\Phi(t_{k+1}, t_{k+L-1}) \\ \mathbf{0}_{n_z\times n_x} & \mathbf{0}_{n_z\times n_x} & \cdots & \mathbf{H}_{k+1}\Phi(t_{k+2}, t_{k+L-2}) & \mathbf{H}_{k+1}\Phi(t_{k+2}, t_{k+L-1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{n_z\times n_x} & \mathbf{0}_{n_z\times n_x} & \cdots & \mathbf{H}_{k+L-3} & \mathbf{H}_{k+L-3}\Phi(t_{k+L-2}, t_{k+L-1}) \\ \mathbf{0}_{n_z\times n_x} & \mathbf{0}_{n_z\times n_x} & \cdots & \mathbf{0}_{n_z\times n_x} & \mathbf{H}_{k+L-2} \\ \mathbf{0}_{n_z\times n_x} & \mathbf{0}_{n_z\times n_x} & \cdots & \mathbf{0}_{n_z\times n_x} & \mathbf{0}_{n_z\times n_x} \end{bmatrix} \tag{53}$$

| NORAD-ID | 55131 | 59773 |
|---|---|---|
| Observation window | 21 days | 11 days |
| Number of observations | 30 | 30 |
| Prediction window | 6 days | 3.5 days |
| Number of states for prediction | 10 | 10 |
| Observation noise $\sigma_r$ [m], $\sigma_v$ [m/s] | 300, 0.05 | 500, 0.1 |
| ASNC window length | 3 | 3 |

*Table 1. Test case description. The observation noise has been defined via a trial and error procedure, but in general the problem is sensitive to within half an order of magnitude. Also recall the NEGA method does not make use of the above observation noise.*

## 4. RESULTS

The approaches described in Sections 3.2 and 3.3 have been evaluated for the test cases defined in Table 4[1]. TLEs were downloaded from space-track.org for both the observation window and prediction window, obtaining the number of observations and number of states for prediction performance evaluation displayed in the table. Note for implementation purposes, the TLEs are converted to J2000 Cartesian coordinates, i.e. propagated with a null time delta, in order to be processed. The three selected objects are a US Wide Area Augmentation System (WAAS) satellite, (Galaxy 30 with NORAD-ID 46114), a Chinese GEO telecommunications satellite (SHIJIAN-23 with NORAD-ID 55131) and a Russian satellite in Sun Synchronous Orbit (SSO) (COSMOS 2576 with NORAD-ID 59773).

The methods are evaluated in terms of prediction error and covariance containment, this is, the accuracy of state prediction compared to the reference and the Mahalanobis distance of the difference between the predicted and observed states considering the estimated process noise level.

Results for object 55131 using ASNC are summarized in Figures 1 and 2, which include the position estimation distance and standard deviation in the Radial-Transversal-Normal (RTN) body-fixed reference frame, and the Mahalanobis distance, respectively. Position prediction error in the RTN frame shows a rather consistent behavior, with the standard deviation of the predicted position covariance building over time in a similar fashion to the position differences. The radial uncertainty grows at a moderate rate, and the standard deviation in the normal direction builds up slowly over time. Uncertainty in the transversal or in-track component grows significantly faster, something to be expected as it relates to the actual location of the satellite within its (perturbed) elliptical motion around the Earth. While the number of samples may be deemed insufficient for a proper statistical analysis, it can be seen that the Mahalanobis distance of

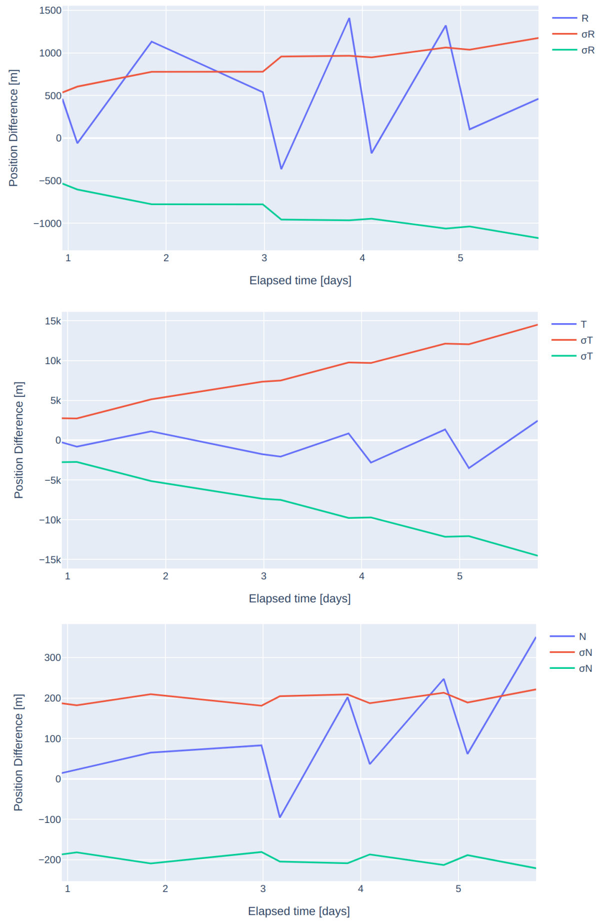


*Figure 1. Position difference of predicted versus observed (TLE) values in the radial, tangential and normal direction for object 55131 using ASNC.*

the reference states with respect to the predicted Gaussian PDF is kept lower than 3, a value that is commonly used for covariance containment analysis [13].

The results of the evaluation of the NEGA method using the TLE sequence for object 55131 can be consulted in Figures 3 and 4. In this case, the filter was able to jointly estimate the process noise and the measurement noise, the latter corresponding to $\sigma_r = 605.4$ [m] and $\sigma_v = 0.605$ [m/s]. These values are somewhat similar to the ones provided in Table 4 but there seems to be a coupling between the position and velocity process noise values. This coupling could arise from the approach used to solve Equation (50): we pose it in the form of a (weighted) constrained Quadratic Programming (QP) problem for the diagonal entries of $\mathbf{Q}$ and $\mathbf{R}$. Therefore, there are three groups of quantities with very different expected orders of magnitude since $\mathcal{O}(Q) \sim 10^{-10}$, which could possibly affect the solution returned by the QP solver. In terms of absolute prediction error, the results are comparable to those of ASNC, although the consistency exhibited by the predicted states is worse. This is notably the case for the radial position component, for which the estimated process noise level seems unable to

[1]Due to time constraints and perhaps the higher complexity of the problem, results for the NEGA method are limited to the GEO test case.

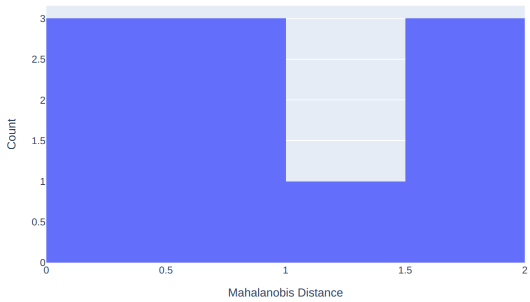


*Figure 2. Histogram of the Mahalanobis distance within the prediction window for object 55131 using ASNC.*

properly contain the observed scatter. Further analysis regarding the QP solver used and the way the problem has been scaled could shed more light for the applicability of the NEGA method, which could in any case be used to obtain educated guesses for the process and observation noise levels.

The results for object 59773 using ASNC are summarized in Figures 5-6. In this case, the estimated process noise closely follows the state prediction error for the time window of analysis. It is clear that the error committed while estimating the state is high but dynamics in LEO are known to be less predictable due to the effect of atmospheric drag.

## 5. CONCLUSIONS

The work conducted in this paper explores operationally viable methods to perform uncertainty quantification for orbital state prediction based on state vector time series with unknown noise characteristics. The first method, based on covariance matching and known observation noise, has shown to be robust and consistent for the test cases presented, yet it required offline tuning of the expected position and velocity uncertainties. The NEGA estimator, on the other end, appears to be capable of jointly estimating the process and measurement noise, discerning the order of magnitude of the different uncertainty sources. Nonetheless, it appears to be outperformed by a moderately tuned ASNC in terms of estimation accuracy and covariance containment. It shall be emphasized that the maturity of the NEGA implementation is relatively low so further research could improve the observed results.

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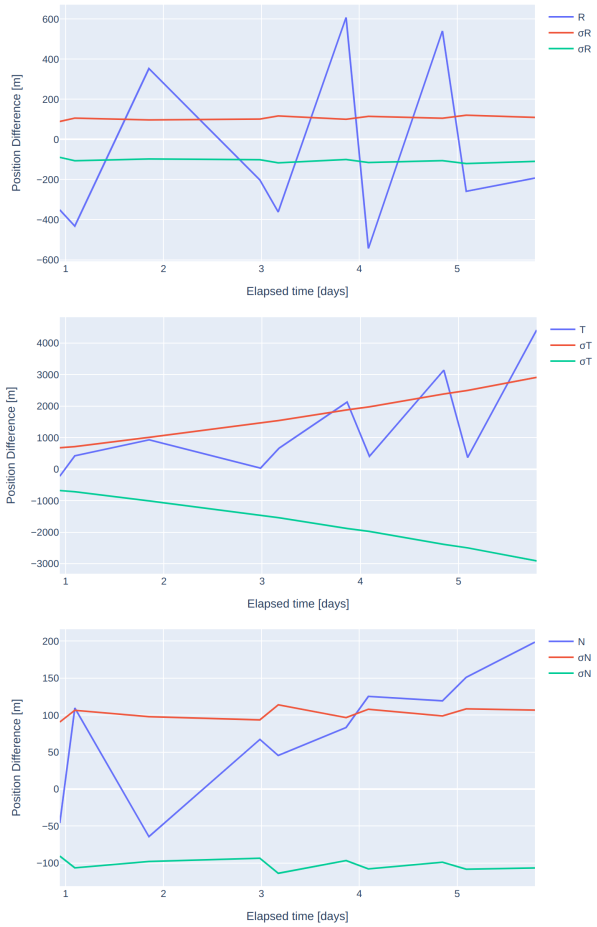


*Figure 3. Position difference of predicted versus observed (TLE) values in the radial, tangential and normal direction for object 55131 using NEGA.*

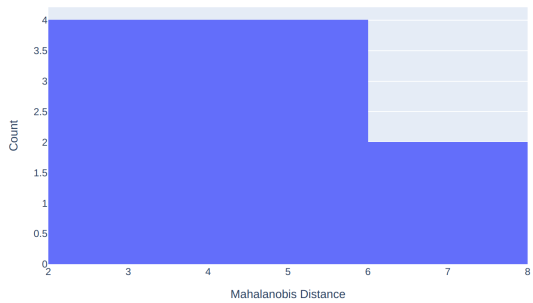


*Figure 4. Histogram of the Mahalanobis distance within the prediction window for object 55131 using NEGA.*

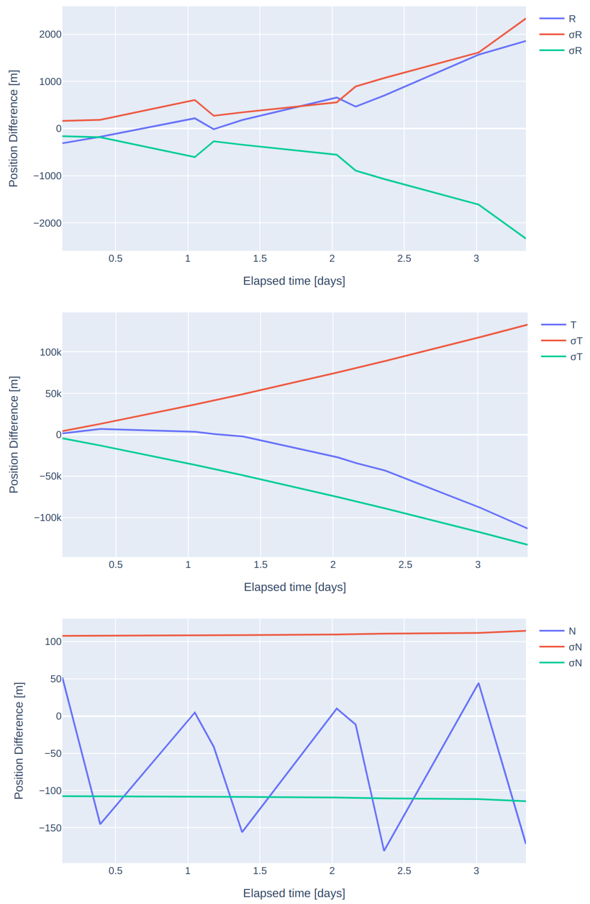


*Figure 5. Position difference of predicted versus observed (TLE) values in the radial, tangential and normal direction for object 59773 using ASNC.*

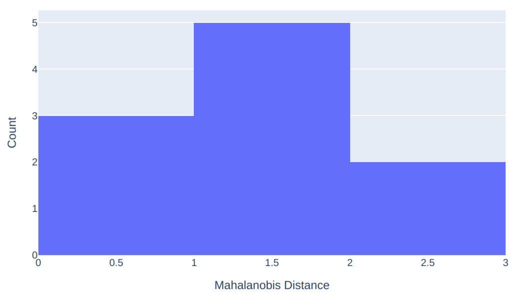


*Figure 6. Histogram of the Mahalanobis distance within the prediction window for object 55973 using ASNC.*

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