GAUSSIAN MIXTURE-UNSCENTED TRANSFORM PROPAGATION DURING RE-ENTRY FRAGMENTATION FOR IMPACT UNCERTAINTY QUANTIFICATION

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ABSTRACT

Motivated by the ever-increasing space debris population in LEO and subsequent re-entry events, risk analysis of the effects of uncontrolled re-entry on-ground is considered a key component of modern space debris mitigation strategies.

A standard approach of quantifying the uncertainty associated with re-entry makes significant use of Monte Carlo (MC) campaigns but the heavy computation burden of invoking MC means that physical models applied to re-entry modelling in terms of aerothermodynamics, structures and flight dynamics must often be necessarily limited. This work proposes the use of nonlinear propagation techniques commonly applied in on-orbit uncertainty quantification as a means to acquire results without needing to expend significant resources on an MC campaign.

The Unscented Transform (UT) significantly reduces the requisite function evaluations necessary to propagate a probability distribution through a nonlinear function. This is achieved by sacrificing information on the distribution beyond the first two statistical moments. If a distribution can be safely assumed (almost always as a normal distribution), the UT presents significant computational savings over MC. Unfortunately, re-entry processes can cause positional uncertainty to deviate significantly from Gaussian descriptions, thus the motivation for applying a Gaussian Mixture Model (GMM) to approximate a non-Gaussian distribution as a weighted sum of a "library" of Gaussians. These two propagation techniques together comprise the hybrid model known as GMM-UT.

The common use of covariance matrices in orbital uncertainty propagation enables interoperability between standard orbital methods and the proposed re-entry uncertainty propagation. Additionally, the lower computational expense provides the opportunity to utilise more sophisticated physical models for re-entry, be that in terms of higher fidelity aerothermodynamical computation, advanced structural models or in this case where a larger quantity of fragments and fragmentation events than would usually be feasible for MC propagation can be analysed.

In this work, the comparative cheapness of GMM-UT is leveraged to propagate a cluster of spacecraft fragments simultaneously whilst still computing the aerodynamics and aero-induced heating on these bodies. This is performed using the TransatmospherIc flighT simulAtioN tool (TITAN), a code developed by the University of Strathclyde for the purpose of simulating hypersonic re-entries utilising models of multiple fidelities.

TITAN's panel-code models approximate aerodynamic effects enabling efficient propagation of multivariate uncertain initial conditions in position and velocity. This is done whilst accounting for both purely translational (6 dimensional) and combined translational and rotational uncertainty (13 dimensional).

Results obtained for a more trivial fragmentation case are compared, utilising statistical similarity measures, against conventional Monte Carlo campaigns and single-distribution Unscented Transform propagation. They are used alongside the results obtained for a more complex case to advocate for further exploration of nonlinear propagation as a method for efficient uncertainty quantification during re-entry.

Keywords: Re-entry; Uncertainty Quantification; Nonlinear propagation; Gaussian Mixture; Space Debris.

1. INTRODUCTION

Space debris in the Low Earth Orbit (LEO) environment is a pressing issue for future space sustainability that cannot be ignored but a hollistic approach to space sustainability cannot only consider the effects of space debris on-orbit, debris objects from LEO are re-entering the atmopshere and impacting Earth at an increasing rate[6], thus the desire for performant analysis tools and methodologies to appropriately assess the impact of space-debris on ground in a robust and uncertainty-aware manner.

This work seeks to apply a method frequently invoked in on-orbit uncertainty quantification (UQ) and uncer-

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tainty propagation (UP), namely the Unscented Transform (UT)[4], to propagate uncertainty associated with the state vector of re-entering debris to ground for the purpose of appropriately quantifying the resultant impact distribution of surviving fragments. Standard practices of Monte Carlo (MC) campaigns for UQ carry a heavy computational burden that the UT seeks to avoid.

The TransatmospherIc flighT simulAtioN tool (TITAN) [7]

2. NONLINEAR PROPAGATION

The problem sought to be solved in dynamical uncertainty propagation is to propagate a state vector $\vec{\mathbf{x}}_i \in \mathbb{R}^d$ with some initial probabilistic description $p_i(\vec{\mathbf{x}}_i)$ through a function $f(\vec{\mathbf{x}}_i) \in \mathbb{R}^d$ in order to recover the resultant distribution $p(f(\vec{\mathbf{x}}_i))$, i.e. $p_{i+1}(\vec{\mathbf{x}}_{i+1})$. Note that in order for the result that p_{i+1} can be described by a linear transformation of p_i it is required that p_i is a Gaussian distribution and that $f(\vec{\mathbf{x}})$ can be assumed to be a linear function on $\vec{\mathbf{x}}$. This second requirement creates problems as it is intuitively not the case for a great many dynamical systems and especially systems with complex aerodynamics such as transatmospheric re-entry.

Thus methods must be explored which seek to recover p_{i+1} with repeated propagations of system dynamics, classically Monte Carlo methods are employed to solve this problem.

$$\mathbb{E}[p_{i+1}] \approx \mu_{MC} = \sum_{j=1}^{N_{MC}} \frac{f(\vec{\mathbf{x}}_{i,j})}{N_{MC}} \tag{1}$$

However N_{MC} must be a sufficiently large number in order for MC to appropriately approximate the true value. Julier and Uhlmann showed[4] that by restricting the problem to only recover the first two statistical moments and selecting points appropriately, the necessary number of samples is far below the number required for Monte Carlo convergence ($N_{UT} = 2 \cdot d + 1 << N_{MC}$). This so-called Unscented Transform (UT) in practice means assuming an unchanging distribution kernel with varying mean and (co) variance, which is almost always assumed to be a Normal distribution. The points selected to propagate for an Unscented Transformation, termed *sigma points*, are obtained deterministically[5].

$$\begin{aligned} \vec{\mathbf{x}}_{i,\sigma_1} &= \vec{\mathbf{x}}_{i,\mu} \\ \vec{\mathbf{x}}_{i,\sigma_j} &= \begin{cases} \vec{\mathbf{x}}_{i,\mu} + \sqrt{\lambda + d} \cdot \sqrt{\Sigma_k} & \text{if } j < d+1 \\ \vec{\mathbf{x}}_{i,\mu} - \sqrt{\lambda + d} \cdot \sqrt{\Sigma_k} & \text{if } j \geq d+1 \end{cases} \end{aligned} \tag{2}$$

Where $\sqrt{\Sigma_k},\ k \in \{1, \dots, d\}$ is the k^{th} column of the square-root covariance matrix and $\lambda = \alpha^2(d+\kappa) - d$ with α and κ as scaling parameters. The mean and covariance can then be reconstructed according to weighted sums of the propagated sigma points

$$\mathbb{E}[p_{i+1}] \approx \mu_{UT} = \sum_{j=1}^{2d+1} w_{\mu,\sigma_j} f(\vec{\mathbf{x}}_{i,\sigma_j}) \tag{3}$$

$$\Sigma[p_{i+1}] \approx \sum_{j=1}^{2d+1} w_{\Sigma,\sigma_j} [f(\vec{\mathbf{x}}_{i,\sigma_j}) - \mu_{UT}]^2 \tag{4}$$

Where $w_{\mu,\sigma_j} = w_{\Sigma,\sigma_j} = 1/[2(n+\lambda)] \quad \forall j > 1$ and $w_{\Sigma,\sigma_1} = \lambda/(n+\lambda), \quad w_{\mu,\sigma_1} = w_{\Sigma,\sigma_1} + (1 - \alpha^2 + \beta)$ with β as a term for incorporating prior knowledge (for a Gaussian prior $\beta = 2$ is optimal).

The UT is a desirable method in a variety of contexts where the system dynamics can be assumed to be linear over a small timestep, thus by assuming a Gaussian initial case on can effectively propagate a transformed distribution. Unfortunately in orbital (and indeed suborbital) contexts a well-behaved initial Gaussian can be observed to distort into a "banana" distribution over time.

2.1. Gaussian Mixture Models

To accurately capture distributions which deviate from Gaussian representations over time, a Gaussian Mixture Model (GMM) is invoked. Any distribution can be approximated as a weighted sum of Gaussians[3].

$$p \approx p_{GMM} = \sum_{n=1}^{N_G} w_n \mathcal{N}(\mu_n, \Sigma_n) \tag{5}$$

It should be noted that unless the GMM is changed during the propagation that the assumption that system dynamics can be linearly approximated over short timescales is still necessary.

Satellites and space debris can be readily characterised with a covariance ellipsoid describing the uncertainty in their orbital elements. This is the motivation for selecting a multivariate Gaussian in state space as the initial uncertainty that is propagated through re-entry. Thus the GMM can be defined as a collection of recursively split Gaussians. Libraries of component size 3 and 5 respectively, determined by DeMars[2], are applied to appropriately split the Gaussians according to the following splitting scheme in the direction of the m^{th} principal axis of the covariance, usually taken to be the largest.

$$\begin{aligned} w_n &= w_0 W_n \\ \vec{\mu}_n &= \vec{\mu}_0 + \sqrt{\lambda_m} M_n \vec{\mathbf{e}}_m \\ \Lambda_n &= \text{Diag}[\lambda_1, \dots, s_n^2 \lambda_m, \dots, \lambda_d] \\ \Sigma_n &= E \Lambda_n E^T \end{aligned} \tag{6}$$

Where λ_m is the eigenvalue corresponding to the m^{th} eigenvector ($\vec{\mathbf{e}}_m$), E is the eigenvector matrix and W_n, M_n and s_n are the splitting parameters determined by the library.

Once the GMM is split appropriately it can then be propagated on a per-Gaussian basis, before perfoming an Unscented Transform. It should be noted that methods[2] exist to adaptively split a GMM in response to system nonlinearity, something that is desirable but non-trivial in this precise context. Future work on this methodology should likely consider this direction.

3. DYNAMICS DURING RE-ENTRY

Whilst in the orbital regime attitude can, in many cases, be decoupled from state dynamics this is not true for descent trajectories, especially for the continuum flow regime and unstable uncontrolled re-entries. To propagate the translational and rotational state vectors together necessitates a 13-dimensional parameter space of 3 translational positions, 3 translational velocities, 4 rotational positions in the form of a quaternion (to avoid gimbal lock) and 3 rotational velocities.

$$\vec{\mathbf{x}} = \begin{cases} \vec{\mathbf{r}} \\ \dot{\vec{\mathbf{r}}} \\ q \\ \omega \end{cases} \in \mathbb{R}^{13} \tag{7}$$

Propagation of a 13-dimensional state vector obviously increases the computational effort required to capture the statistics of the problem and this means that often a trade-off between physical accuracy and statistical convergence must be made. In this context TITAN enables the use of a variety of numerical integration schemes, applied in this context is a 3rd order Runge-Kutta method, TITAN's aerothermodynamic modelling capabilites enable computation of aerodyanmic forces throughout all regimes experienced during a re-entry.

3.1. Aerothermodynamic Modelling

The **T**ransatmospher**I**c fligh**T** simul**A**tio**N** tool (TITAN) enables the modelling of aerodynamical and aerothermodynamical effects in the free-molecular and continuum hypersonic flow regimes at varying levels of fidelity, notably one can select between panel-based impact methods implemented according to Modified Newtonian Theory or Schaaf-Chambre[8] as opposed to full-scale volumetric fluid analysis in the form of non-equilibrium computational fluid dynamics (CFD) or Direct Simulation Monte Carlo (DSMC). In this context the application of the more performant (lower fidelity) panel methods as the physical model within the GMM-UT propagation enables the analysis of a larger number of fragments than could be analysed with Monte Carlo propagation.

$$c_p = \begin{cases} c_{p_c} & \text{where } K_n < 10^{-3} \\ c_{p_f} & \text{where } K_n > 10^{2} \\ f_{bridging}(c_{p_c}, c_{p_c}) & \text{where } 10^{-3} < K_n < 10^{2} \end{cases} \tag{8}$$

4. APPLICATION TO RE-ENTRY

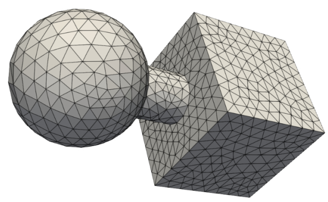
In order to assess the viability of the GMM-UT methodology when applied to the re-entry UQ problem this work considers two distinct test cases with similar initial uncertainty (Table 1), described as multivariate normal distributions in 12-dimensional state space, (with attitude specified by Euler angles as opposed to quaternions).

Table 1: Uncertain input parameters

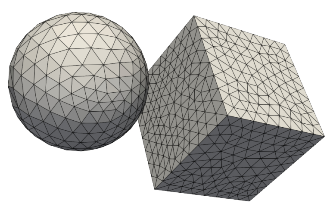
Parameter (units)	Mean	Variance
X Position (m)	6498.1×10^3	1690.0×10^3
Y Position (m)	0.0	250.0×10^3
Z Position (m)	0.0	250.0×10^3
X Velocity (m/s)	-344.59	6.25
Y Velocity (m/s)	0.0	25.0
Z Velocity (m/s)	7892.48	25.0
Roll Position (rad)	0.0	0.44
Pitch Position (rad)	0.0	0.44
Yaw Position (rad)	0.0	0.44
Roll Velocity (rad/s)	0.0	0.0044
Pitch Velocity (rad/s)	0.0	0.0044
Yad Velocity (rad/s)	0.0	0.0044

4.1. Simple Geometry

In order to assess the described methodology in a re-entry context it is applied to a simplistic approximation of the kinds of objects that re-enter from low Earth orbit, namely a sphere joined to a cube by a demisable joint specified to demise at $h_{\text{frag}} = 78$ km (Fig. 1). The sphere and cube provide a an opportunity to assess the effects of rotational uncertainty on re-entry, as the aerodynamic forces acting upon the sphere are invariant with attitude. The simplicity of this representation means results can be directly compared with a Monte Carlo campaign.



(a) Mesh prior to fragmentation event



(b) Mesh after fragmentation event

Figure 1: The simple geometry before (a) and after (b) fragmentation

4.2. Comparison With Monte Carlo

Fig. 2 shows the altitude over time across 2456 Monte Carlo runs, the dispersive effects of attitude uncertainty on a non-axisymmetric body can be observed. In Fig. 3 it can be seen that whilst a 3-DoF GMM-UT cannot capture these effects the 6-DoF GMM-UT gives an acceptable representation of the statistics of the problem. It is made clear by this comparison that the 3 degree of freedom model cannot be expected to resolve the resultant statistics of the problems effectively, thus results for the advanced case were obtained with the 6-degree of freedom model.

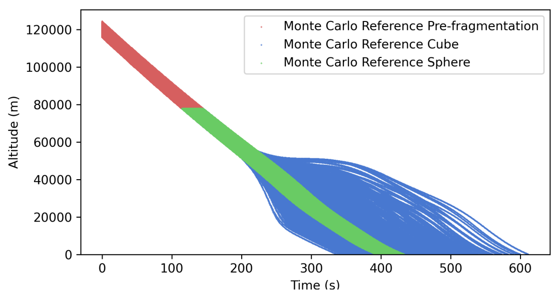


Figure 2: Altitude vs Time plot of the 2456 reference Monte Carlo Runs

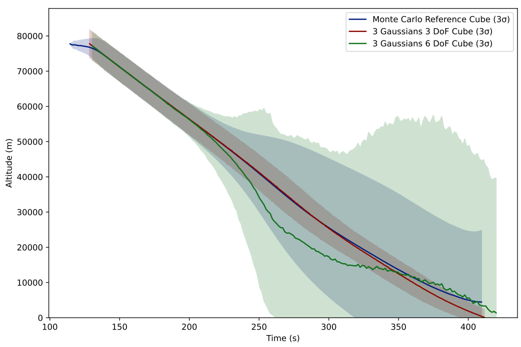


Figure 3: Comparison of 3-DoF and 6-DoF GMM-UT to reference Monte Carlo

4.3. Advanced Case

Many instances of 2nd stage DELTA-II rocket bodies are remain in low, decaying orbits above the Earth, this makes them important and desirable targets for analysis. One particular re-entry event lead to the recovery of fragments on-ground[1], enabling direct comparison of re-entry simulation to reality. It should be noted that not only is the available orbital state information uncertain in nature, in that it is provided in the form of a Two/Three Line Element (TLE) which has an associated error, but the attitude of the object at entry interface can be considered to be completey unknown. Thus this case was considered an apt “stress-test” in this context.

Fig. 4 shows the mesh used in the TITAN simulation of the DELTA-II R/B where a 6-DoF 3-Gaussian GMM was applied to the problem as the most computationally efficient method explored in this work. Fig. 5 shows 2500 samples of the GMM trajectory of the object drawn at each timestep and FIG shows selected slices of the state vector distribution across altitude.

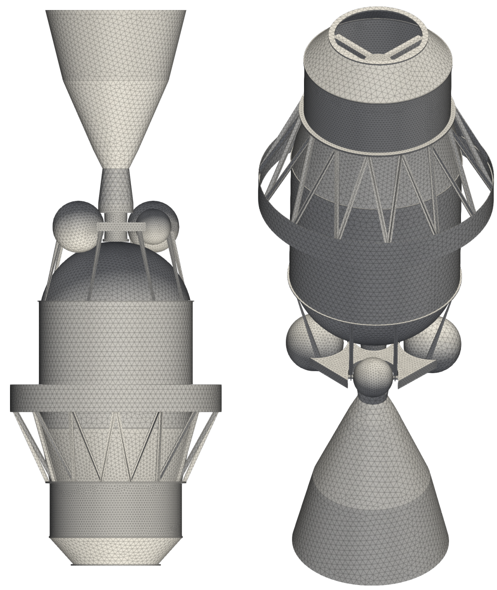
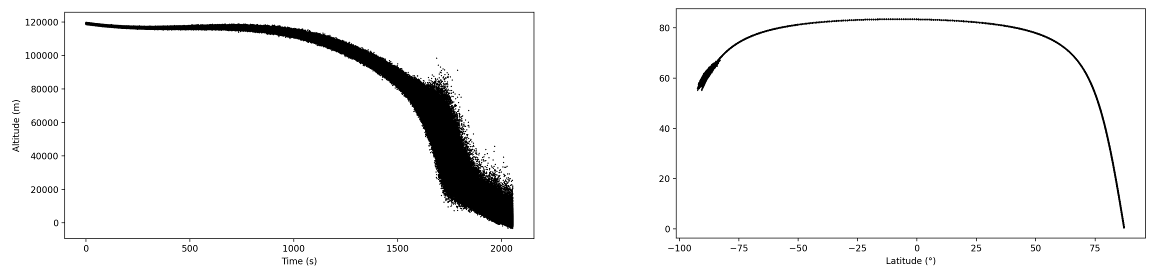


Figure 4: The DELTA-II geometry used in analysis

It can be seen from these figures that the distribution of position and velocity can be highly non-Gaussian for complex re-entry cases

5. CONCLUSION

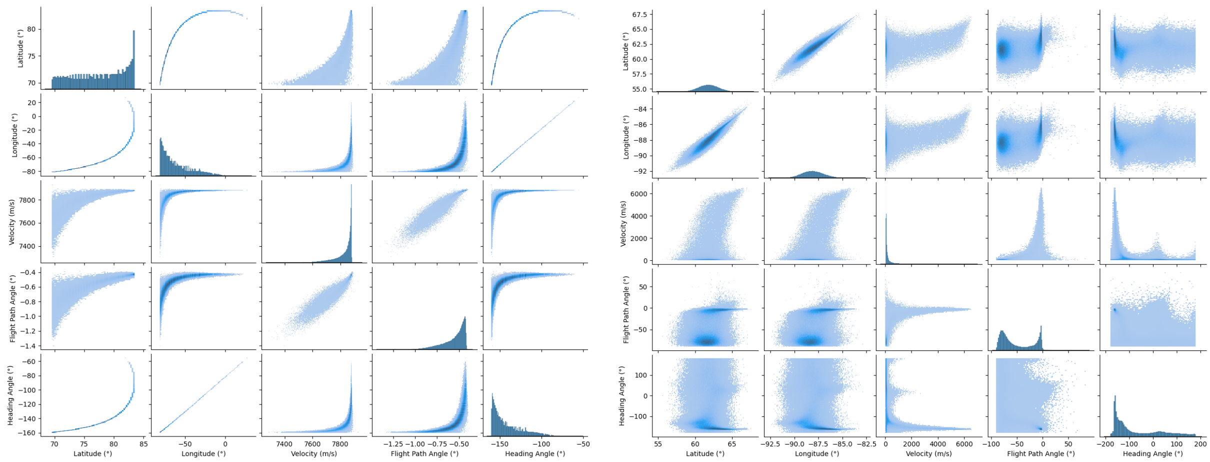
In this work the GMM-UT method was applied at first to a simplified test case in order to assess the performance of differing implementations before stress-testing the proposed method on the more complex test case of the Delta-II. It can be concluded that GMM-UT can be applied to re-entry uncertainty quantification contexts although the methodology is not without its challenges. It can be seen that 3 degree of freedom representations have issues accurately capturing the dynamics of the system. As system dynamics become higher-dimensional and less linear the advantages presented by this method decrease. Nevertheless the computational savings open new avenues that would not be feasible with traditional MC methods, for example in simulating expensive re-entry events with many objects.



(a) Altitude vs Time of Delta II Trajectory

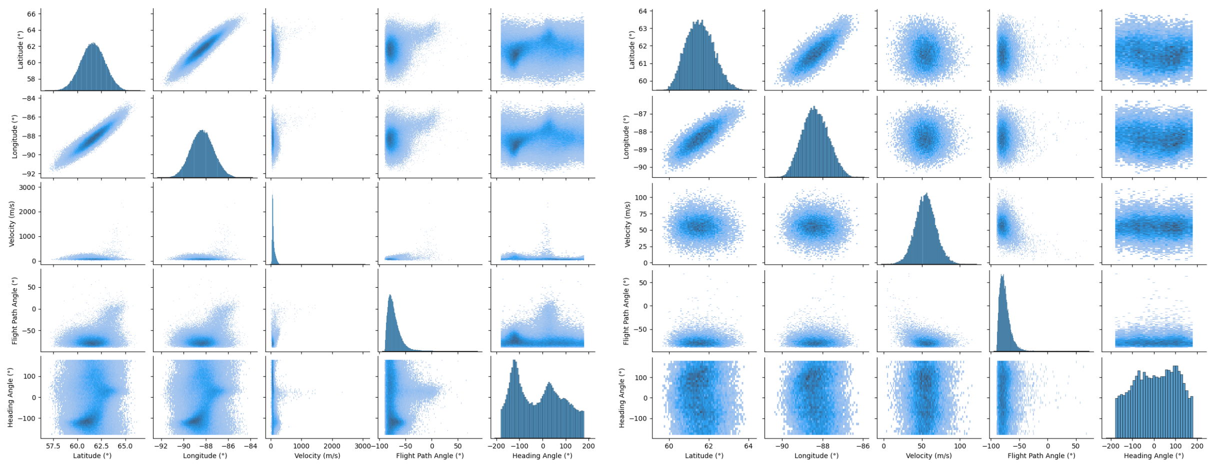
(b) Ground Track of Delta II Trajectory

Figure 5: Samplings of the GMM-UT applied in this context give an idea of the envelope of fragment positions



(a) State distribution at an altitude of 100km

(b) State distribution at an altitude of 60km



(c) State distribution at an altitude of 20km

(d) State distribution at an altitude of 1km

Figure 6: "Slices" of the Delta-II state probability at at selected constant altitudes of 100km (a), 60km (b), 20km (c) and 1km (d)

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