ON THE ARNOLD DIFFUSION MECHANISM IN MEO: A SUMMARY

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ABSTRACT

In this work, we summarize the procedure developed in [1, 2] to justify rigorously the existence of Arnold diffusion for satellites in Medium Orbit Orbit, that are affected by the perturbation due to the Earth's oblateness and the gravitational acceleration exerted by the Moon. The motivation is the need of designing affordable end-of-life solutions also from high-altitude orbits. These can be conceived not only on the basis of novel technologies, but also following an advanced theoretical understanding. The hypothesis of the work presented is that the Arnold diffusion process is the cause of the eccentricity growth found numerically by previous works, that is thus not due to the chaotic behavior due to overlapping resonances. The theoretical outcome can also explain the phase dependence previously found.

Keywords: Medium Earth Orbit; disposal; third-body perturbation; Arnold diffusion.

1. INTRODUCTION

In the last 20 years, since the work of Jenkin and Gick [3] a large effort (e.g., [4]-[10]) has been devoted to analyze the eccentricity growth mechanism in Medium Earth Orbit (MEO) and its practical implications. The long-term eccentricity variation due to the third-body perturbation is well known since the '60s (e.g., [11]), but only recently its value has been recognized, to achieve quasi-natural transfers, notably, towards an Earth's reentry for satellites in MEO. In this context, the numerical simulations performed in the past showed that it is possible to obtain a natural eccentricity growth as large as to get into the atmospheric domain, but this growth cannot be justified within a pure Lidov-Kozai mechanism, that takes place, but it is not sufficient to achieve the values of eccentricity required to reenter, assuming the nominal value of inclination of the Global Navigation Satellite System (GNSS) constellations.

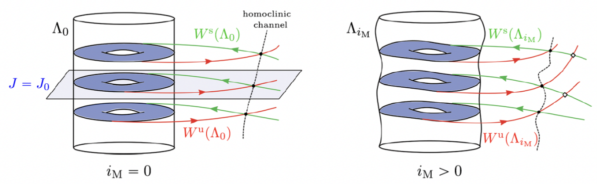

Figure 1. When the inclination of the Moon i_M is assumed to be 0, the system is autonomous and the spacecraft can move only within a given energy level (left). When $i_M > 0$, we can move to a different energy level thanks to homoclinic connections occurring along the normally hyperbolic invariant manifolds. The figure is taken from [1].

In this work, we summarize our effort, well detailed in [1, 2], to provide a well-structured theoretical framework for the so-called Arnold diffusion mechanism, first proposed in [9, 10], that can lead to such a high value of eccentricity. Assuming that the satellite is affected by the Earth's monopole, the Earth's oblateness and the gravitational perturbation exerted by the Moon, the reentry is achieved not because of a chaotic behavior due to overlapping resonances as suggested in [7], rather by the fact that the orbit of the Moon is inclined with respect to the ecliptic plane and thus the dynamical system is non-autonomous, allowing for jumps between different energy levels.

We will take as example the orbit of the Galileo constellation, but the procedure is general.

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2. BASIC FRAMEWORK

The basic theoretical ingredients are the following.

- In quasi-integrable Hamiltonian systems, Arnold diffusion consists in the slow variations of the actions of the system due to the accumulation of small perturbations over time. In the case considered in this work, we have also a variation in energy because the system is non-autonomous.
- For our problem, the action is the orbital eccentricity of the satellite, that changes by 'drifting along a resonance'.
- It is well known that the inclination of the Galileo constellation, $i \approx 56°$, corresponds to the resonance $2\dot{\omega} + \dot{\Omega} = 0$, where i, the argument of pericenter ω and the longitude of the ascending node Ω are assumed with respect to the Earth's equatorial plane.
- Assuming that the orbit of the Moon is coplanar with the one of the Earth (that is, $i_M = 0$), the system is autonomous and has dimension 4 (2-degree of freedom).
- In the neighborhood of the resonance, that is, an equilibrium point, there exists a family of hyperbolic periodic orbits (each one characterized by a different value of energy), and thus a stable and an unstable invariant manifold stemming from each of them.
- Along the hyperbolic manifolds, assuming that the associated periodic orbit corresponds to a circular one (that is, $e = 0$) the eccentricity grows naturally up to a certain value (Lidov-Kozai mechanism). However, for the specific orbital parameters of Galileo, that is not sufficient to achieve reentry.
- These hyperbolic invariant manifolds intersect transversally, forming a homoclinic orbit associated with the given energy level.
- In the extended phase space (adding time) the hyperbolic periodic invariant orbits become 2-dimensional invariant tori. Its union form a so-called *normally hyperbolic invariant cylinder* (see Fig. 1 - left), which has unstable and unstable invariant manifolds that intersect transversally.
- When the inclination of the Moon is taken into account ($i_M > 0$), the system becomes non-autonomous due to the longitude of the ascending node of the Moon. The normally hyperbolic invariant cylinder is persistent (for $i_M > 0$ small enough). Inside the cylinder, most of the hyperbolic tori persist (thanks to KAM theory [12]). The invariant manifolds of the cylinder are also robust.
- When $i_M > 0$, the hyperbolic invariant manifolds stemming from the invariant tori in the cylinder also intersect transversally, but this time they can connect also with hyperbolic invariant manifolds stemming from tori at different energy levels (see Fig. 1 - right).
- Assuming that the inclination of the Moon is small ($i_M \approx 5.15° \approx 0.08$ rad with respect to the ecliptic plane), the stable and unstable invariant manifolds of the cylinder can be studied perturbatively, by means of the so-called Poincaré-Melnikov Theory [13]). It describes how homoclinic orbits to the cylinder may be heteroclinic between different tori.

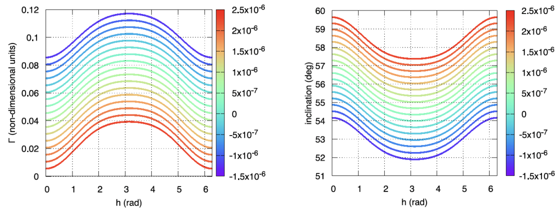

Figure 2. Examples of periodic orbits for $\tilde{\mathrm{H}}_{CP,1}$ assuming $a = 29600$ km (Galileo semi-major axis) and the eccentricity to be 0. Non-dimensional units. On the right, the inclination along the periodic orbit, as a function of h. The colorbar reports the value of the energy, the one of Galileo being equal to 0. The figure is taken from [2].

More details are given in the following section.

3. MAIN EQUATIONS AND INVARIANT OBJECTS

Let us consider Delaunay variables, namely,

$$\begin{aligned} L &= \sqrt{\mu a}, & l &= M, \\ G &= L\sqrt{1-e^2}, & g &= \omega, \\ H &= G\cos i, & h &= \Omega. \end{aligned} \tag{1}$$

and let $\alpha = a/a_M$, where a is the semi-major axis of Galileo and a_M the one of the Moon. Thus, the Hamiltonian of the system can be written as

$$\mathrm{H}(L, G, H, g, h, \Omega_M; i_M) = \mathrm{H}_K(L) + \tilde{\mathrm{H}}_0(L, G, H) + \alpha^3 \tilde{\mathrm{H}}_1(L, G, H, g, h, \Omega_M; i_M), \tag{2}$$

where the first term is the Keplerian contribution, the second one is the perturbation due to the Earth's oblateness and the last one is the doubly-averaged quadrupolar third-body perturbation. Since H does not depend on l, the semi-major axis does not change. Moreover, we can notice that the oblateness effect is responsible only of the variation in (g, h), i.e., (ω, Ω), as it depends only on (G, H). On the other hand, the third-body effect can change all the orbital elements (except L and thus a), and it is the sum of different periodic terms that depend on (h, g, Ω_M). We do not show here the whole expansion

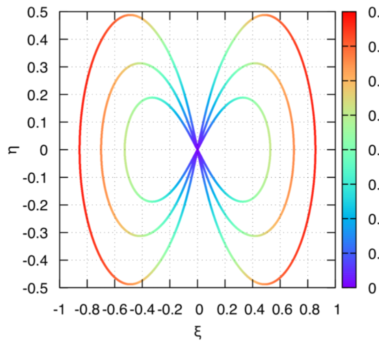

Figure 3. Examples of hyperbolic invariant manifold associated with circular orbits on the Poincaré section $h = 0$. The colorbar reports the maximum eccentricity growth. The units used are non-dimensional. The variables are Poincaré variables to avoid issues due to the fact that we are considering $e = 0$. The figure is taken from [1].

that can be found for instance in [7]. If $i_M = 0$, it can be proven that the Hamiltonian does not depend on Ω_M, and thus it is autonomous.

As a first approximation we assume that, for the dynamics of Galileo, the dominant term in $\tilde{\mathrm{H}}_1(L, G, H, g, h, \Omega_M; i_M)$ is the one depending on $2\omega + \Omega$ since, as mentioned before, $2\dot{\omega} + \dot{\Omega} \approx 0$ at the Galileo constellation and thus the corresponding cos term is almost constant. Moreover, $\tilde{\mathrm{H}}_1$ can be split in the sum of an autonomous and non-autonomous part, namely,

$$\tilde{\mathrm{H}}_1 = \tilde{\mathrm{H}}_{CP,1}(i_M = 0) + \tilde{\mathrm{R}}(i_M \neq 0). \tag{3}$$

Notice that both terms depend on all periodic terms, not only on $2\omega + \Omega$.

In the neighborhood of the resonance, we can distinguish two timescales corresponding to the third-body perturbation: a slow one, corresponding to the resonant angle, and a fast one corresponding to h. The equilibrium point mentioned in the previous section corresponds to the case when the autonomous part of the third-body perturbation is averaged over the fast variable h, (that is the Hamiltonian $\mathrm{H}_K + \tilde{\mathrm{H}}_0$ plus the part in $\tilde{\mathrm{H}}_{CP,1}(i_M = 0)$ that depends only on $2\omega + \Omega$, see Eq. (2.20) in [2]). Under this approximation, we have an equilibrium point

$$\dot{e} = 0, \qquad 2\dot{\omega} + \dot{\Omega} = 0$$

for each value of the integral of motion $\Gamma = H - G/2$, that is, we can choose as action e or i, as the other one will be determined by Γ.

If, instead, we consider the full coplanar contribution[1], then the hyperbolic equilibrium points become hyperbolic periodic orbits in Poincaré variables (ξ, η, Γ, h), where (ξ, η) are function of $(a, \sqrt{1-e^2}, (2\omega + \Omega)/2)$. A given periodic orbit can be visualized as a function of (Γ, h) or (i, h), as shown in Fig. 2 for $e = 0$. Note that the equilibrium points and the corresponding periodic orbits are hyperbolic for a small set of Γ [2].

[1] The detailed analysis on the difference between the invariant objects of the $h-$averaged coplanar approximation and the full coplanar approximation is given in [2].

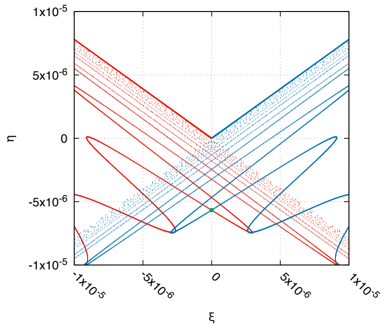

Figure 4. An example of where the splitting angle is computed for the homoclinic connections. The units used are non-dimensional. The variables are Poincaré variables to avoid issues due to the fact that we are close to $e = 0$. The figure is taken from [1].

We then compute numerically the homoclinic connection associated with the given periodic orbit at a given energy level[2], considering a well-defined Poincaré section at $h = 0$ or $h = \pi$, and verify, again numerically, that the connection between the stable and the unstable manifold is transversal, i.e., the splitting angle is not 0. In Fig. 3, we show the behavior of the hyperbolic manifolds for different energy levels on the $h = 0$ Poincaré section and in Poincaré coordinates, showing also the maximum eccentricity that can be achieved (colorbar). Notice that to achieve reentry for Galileo, the eccentricity should reach the value 0.78, that is, we should get to an energy equal to about 1.3×10^{-6} in the non-dimensional units considered, that corresponds to an inclination equal to about 58° for a circular orbit (see Fig. 2). In other words, if Galileo were at this inclination, in principle the Arnold diffusion mechanism would not be required, because along the corresponding hyperbolic manifolds the eccentricity grows naturally up to such value. In Fig. 4, we show an example of where the splitting angle is computed.

Having this, the Poincaré-Melnikov Theory gives information of how the invariant manifolds of different (close-by) hyperbolic tori intersect transversally in the non-autonomous case. This is encoded in the so-called Melnikov function, which is the first order of the jumps in energy that the heteroclinic orbits may undergo. Roughly speaking, by means of Poincaré-Melnikov Theory we obtain a transition chain, that is a sequence of hyperbolic tori connected by transverse heteroclinic orbits. Such structure acts as a "highway" for the drifting orbits. Finally, a shadowing argument provides orbits that follow closely this transition chain leading to the needed drift in

[2] As energy level, we mean the value of the Hamiltonian.

eccentricity.

4. DISCUSSION AND OPEN POINTS

The findings presented require some comments.

Under the assumptions considered, the time required to obtain a high enough eccentricity growth would be long, but considering a realistic value for the inclination of the Moon this time can be reduced considerably. Notice that the realistic case means to compute numerically all the invariant objects in the non-autonomous 5-dimensional case.

Another point worth to remark is the role of the secondary resonance $2h - \Omega_M$, that can influence the energy drift when the two resonances coincide. Under the hypotheses considered in this work, its role is weak and thus it does not break the normally hyperbolic invariant cylinder structure. In the future, this aspect can be re-analyzed without the perturbative approach.

There are two important additional points. In [6], we found that the maximum eccentricity growth in a given time interval (200 years) depends on the initial values of both (Ω, ω) and Ω_M. On the basis of the results in [1, 2], the first dependence corresponds to how close the initial condition is with respect to the hyperbolic periodic orbit stemming from the hyperbolic equilibrium point in the $h-$averaged system. If it is chosen not as close, but rather on an elliptical invariant tori that stems from an invariant libration curve in the neighborhood of an elliptic equilibrium point in the $h-$averaged system, then the eccentricity growth can be very low, null in the limit (see Fig. 4 in [2]). The dependence on Ω_M is instead associated with the jumps in energy that we can achieve by Arnold diffusion. These jumps are function of the frequency of Ω_M[3] (recall the Melnikov function mentioned before and see Lemma 6.11 in [1]).

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[3]Recall that Ω_M has a period of about 18.6 years.

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