APPLICATION OF CLASSICAL AND LIE TRANSFORM METHODS TO ZONAL PERTURBATION IN THE ARTIFICIAL SATELLITE THEORY

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ABSTRACT

A scalable second-order analytical orbit propagator program is being carried out. This analytical orbit propagator combines modern perturbation methods, based on the canonical frame of the Lie transform, and classical perturbation methods in function of orbit types or the requirements needed for a space mission, such as catalog maintenance operations, long period evolution, and so on. As a first step on the validation of part of our orbit propagator, in this work we only consider the perturbation produced by zonal harmonic coefficients in the Earth's gravity potential, so that it is possible to analyze the behaviour of the perturbation methods involved in the corresponding analytical theories.

Key words: LaTeX; General Perturbation methods; Analytical Orbit propagator.

1. INTRODUCTION

The Iridium/Cosmos satellite collision showed, among other things, the necessity of improving and further researching in numerical, analytical and semianalytical orbit prediction methods.

This work has been conducted in the frame of the analytical orbit prediction methods. It is worth noting that in [1], we can read: "Analytic theories will continue to be important for both operations and theoretical development in order to meet future needs." In this sense, a scalable second-order analytical orbit propagator program (AOPP) is being carried out. This AOPP combines modern, based on the canonical frame of the Lie transform, and classical perturbation methods in function of orbit types or the requirements needed for a space mission, such as catalog maintenance operations, long period evolution, and so on.

The elimination of the short period terms is the starting point of most of the analytical theories in the artificial satellite problem [2, 3, 4, 5]. In fact, some of the most useful AOPPs, SGP4[6, 7] and PPT2[8, 9], are derived from the Brouwer-Lyddane theory. However, there are other analytical theories which start simplifying the problem by means of a Lie transform called the elimination of the Parallax [10]. In this work, we investigate other alternatives which arise from the elimination of the Parallax. These are based on first removing the long period terms by means of a second Lie transform called the elimination of the Perigee [11]. It is worth noting that the combination of the aforementioned Lie transforms completely remove the argument of the latitude. These two transformations are carried out in a closed form of eccentricity and inclination by using Hamiltonian formalism and polar-nodal variables. The transformed Hamiltonian has one degree of freedom in the variables $(r, dr/dt)$. This kind of Hamiltonian has been classically named Radial Intermediary.

In our case, this Radial Intermediary can be integrated using three different approaches. Traditionally, to complete the theory and obtain the mean elements similarly to Brouwer, a further reduction is made through Delaunay variables by means of a Lie transform called Delaunay normalization [12], which averages the problem over the mean anomaly. It is worth noting that this transformation is performed in closed form of eccentricity. The second method replaces the time and variable r by the perturbed true anomaly and $1/r$ in the Hamiltonian, such that the equations of motion become a one-dimensional perturbed harmonic oscillator and the Krylov-Bogoliubov-Mitropolsky method [13, 14, 15] can also be used to integrate them. Finally, the Hamilton-Jacobi method provides us with an analytical solution for the second order Radial Intermediary in terms of trigonometric functions for the case of motions with small eccentricities [16]. Our AOPP implements these three integration methods. The analytical expressions and the orbit propagator, coded in C/C++, were performed by the symbolic-numeric environment *MathATESAT* [17].

Finally, we must remark that this AOPP will be integrated in $\mathcal{A}strody^{\mathcal{W}eb}_{\mathcal{T}ools}$ and, thus can be used through a Web Graphical User Interface in real time. For more details, see the project Web Site (http://tastrody.unirioja.es).

In this paper, we illustrate the perturbation techniques involved in the carrying out of the analytical theory and its numerical validation.

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2. RADIAL INTERMEDIARY

In this section, we formulate the mathematical model which describes the motion of a satellite only perturbed by zonal harmonic potential and the perturbation techniques used to remove the long period terms due to the argument of the perigee.

The Hamiltonian for an Earth satellite perturbed by zonal harmonic potential terms in polar-nodal variables $(r, \theta, \nu, R, \Theta, N)$ is given by

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} \\ &\quad + \frac{\mu}{r}\sum_{n\geq 2}^{m} J_n \left(\frac{\alpha}{r}\right)^n P_n(\sin i\, \sin\theta) \end{aligned} \tag{1}$$

where P_n is the Legendre polynomial of degree n, m is the maximum order of the zonal harmonic perturbation being considered, μ is the gravitational constant, α is the equatorial radius of the planet, and J_n are the zonal harmonic coefficients.

Fig. 1 shows the combinations of Lie transforms used to remove the argument of the latitude. These two Lie transformations, the elimination of the Parallax [10] and the elimination of the Perigee [11], are carried out in a closed form of eccentricity and inclination. The transformed Hamiltonian $\mathcal{H}''$, classically known as Radial Intermediary, has one degree of freedom in the variables (r'', R'') but maintains the short period terms, due to the mean anomaly. It is worth noting that these transforma-

$$\begin{gathered} \mathcal{H}(r, \theta, \_, R, \Theta, N) \\ \downarrow \varphi_1 \\ \mathcal{H}'(r', \theta', \_, R', \Theta', N') \\ \downarrow \varphi_2 \\ \mathcal{H}''(r'', \_, \_, R'', \Theta'', N'') \end{gathered}$$

Figure 1. Lie transforms used to remove the long period terms: φ_1 elimination of the Parallax and φ_2 elimination of the Perigee

tions were carried out maintaining the physical parameters in symbolic form so that they could be valid for any gravity field model.

3. INTEGRATION

Finally the integration of the second-order Radial Intermediary can be done using three different approaches: the Lie transform technique, the classical averaging method and the Hamilton-Jacobi method. To briefly illustrate each of these methods we only considered the perturbation produced by the second zonal harmonics. After the parallax and peregee eliminations the transformed Hamiltonian $\mathcal{H}''$ yields

$$\mathcal{H}'' = \mathcal{H}''_0 + \epsilon\mathcal{H}''_1 + \frac{\epsilon^2}{2}\mathcal{H}''_2 \tag{2}$$

where

$$\mathcal{H}''_0 = \frac{1}{2}\left(R''^2 + \frac{\Theta''^2}{r''^2}\right) - \frac{\mu}{r''} \tag{3}$$

$$\mathcal{H}''_1 = \frac{\Theta''^2}{r''^2}\left(\frac{\alpha}{p''}\right)^2 \left[\frac{3}{4}s''^2 - \frac{1}{2}\right] \tag{4}$$

$$\begin{aligned} \mathcal{H}''_2 &= \frac{\Theta''^2}{r''^2}\left(\frac{\alpha}{p''}\right)^4 \left[-\frac{13}{8} + 3s''^2 - \frac{69}{64}s''^4 \right. \\ &\quad \left. + \left(\frac{3}{8} - \frac{3}{8}s''^2 - \frac{15}{64}s''^4\right)(1 - e''^2)\right] \end{aligned} \tag{5}$$

and the values of e'', p'' in function of the polar–nodal variables are

$$e''^2 = 1 - \frac{2p''}{r''} + \frac{p''^2}{r''^2} + \frac{R''^2 p''^2}{\Theta''^2}, \qquad p'' = \frac{\Theta''^2}{\mu} \tag{6}$$

3.1. Delaunay normalization

Clasically, in order to obtain Brouwer-like mean elements [2], a further reduction is made by means of the Delaunay normalization [12].

First the Hamiltonian $\mathcal{H}''$ is expressed in terms of Delaunay variables (l, g, h, L, G, H). It is worth noting that the zero order Hamiltonian is $-nL$ where $n = \mu^2/L^3$ is the mean motion. Next, we apply the Lie transform $\varphi : (l, g, h, L, G, H) \to (l', g', h', L', G', H')$, the so-called Delaunay Normalization, which up to first order reads

$$\mathcal{H}'''_0 = \mathcal{H}''_0 \tag{7}$$

$$\mathcal{H}'''_1 = \mathcal{H}''_1 - \frac{\mu^2}{L'^3}\frac{\partial \mathcal{W}_1}{\partial l'} \tag{8}$$

The Lie method solves Eq. (8) choosing the form of the transformed Hamiltonian; Delaunay Normalization takes it as the average over the fastest angle l', whereas $\mathcal{W}_1$ is computed as

$$\mathcal{W}_1 = \frac{L'^3}{\mu^2}\int (\mathcal{H}''_1 - \mathcal{H}'''_1)\, dl \tag{9}$$

The transformed Hamiltonian $\mathcal{H}'''$ only depends on the momenta L', G' and H', and so therefore the equations of motion are trivially integrable. The momenta L', G'

and H' are constants, whereas the variables l', g' and h' yield

$$l' = \frac{\partial \mathcal{H}'''}{\partial L}(t-t_0) + l'_0 \quad (10)$$

$$g' = \frac{\partial \mathcal{H}'''}{\partial G}(t-t_0) + g'_0 \quad (11)$$

$$h' = \frac{\partial \mathcal{H}'''}{\partial H}(t-t_0) + h'_0 \quad (12)$$

where l'_0, g'_0, h'_0, L'_0, G'_0, H'_0 are the transformed initial conditions at the epoch t_0.

3.2. Krylov-Bogoliubov-Mitropolsky method

The second proposed method is based on classical averaging method. The transformation of the Radial Intermediary to a perturbed harmonic oscillator equation is made by taking into account the following change of variables and time

$$u = \frac{1}{r''} - \frac{1}{p''}, \qquad r''^2 \frac{ds}{dt} = \Theta'', \qquad v = \frac{du}{ds} \quad (13)$$

After several calculations, the equations of motion become

$$\frac{d^2u}{ds^2} + u = \epsilon \left(\mathcal{Q}_1^{0,0} + u\, \mathcal{Q}_1^{1,0} \right) + \frac{\epsilon^2}{2!} \sum_{i,j} u^j v^{2i} \mathcal{Q}_2^{j,2i} \quad (14)$$

where $\mathcal{Q}_n^{i,j}$ depends on the physical parameters and the momenta Θ'' and N'', as well as representing the coefficients of $u^i v^j$ at order n.

The second-order differential equation 14 has been solved by applying the Krylov-Bogoliubov-Mitropolsky method. [**?** **?** 15] include a detailed explanatory description of this process and its applications to the zonal problem of an Earth-like planet.

3.3. Hamilton-Jacobi method

This third method is used in the case of small eccentricities, so Eq. 5 can be rewritten as

$$\mathcal{H}_2''^b = \frac{\Theta''^2}{r''^2} \left(\frac{\alpha}{p''} \right)^4 \left[-\frac{5}{4} + \frac{21}{8} s''^2 - \frac{21}{16} s''^4 \right] \quad (15)$$

Then, the Hamilton-Jacobi equation for $\mathcal{H}''$ yields

$$\frac{1}{2} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 \right] + \mathcal{V}_1 \left(r, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \nu}; \epsilon \right) = h_0 \quad (16)$$

where h_0 is the energy integral, thus

$$\mathcal{V}_1(r, \Theta, N; \epsilon) = -\frac{\mu}{r''} + \epsilon \mathcal{H}_1'' + \frac{\epsilon^2}{2} \mathcal{H}_2''^b \quad (17)$$

In order to solve Eq. 16 a solution in separable variables in the form $S = \Theta_0 \theta + N_0 \nu + \mathcal{W}(r)$ can be tested.

For more details about the solution of Eq. 16 see [16].

4. NUMERICAL VALIDATION

Finally some numerical comparisons are shown in this section.

In order to analyze the behaviour of two of the three aforementioned integration techniques, the Krylov-Bogoliubov-Miropolsky and Delaunay Normalization methods, we make an exploratory data analysis consisting of the study of errors produced by both methods in the zonal case when they are compared with an accurate numerical integration of the original problem.

Fig. 2 shows a pie chart depicting the distribution of the more than $14,000$ TLEs considerer in this study. Although it is well known that TLEs have been designed to be used in combination with the SGP4 orbit propagator, we consider that a TLE space catalog contains a large and representative number of different types of orbits, which can be considered as a reliable and independent test for our study.

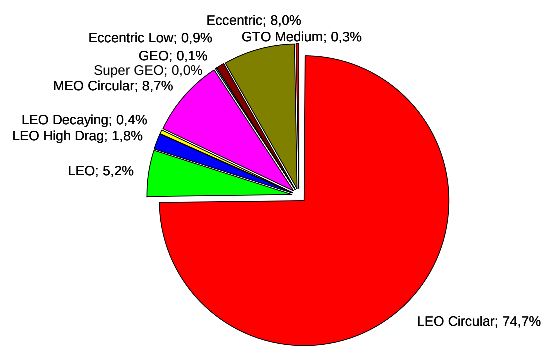

Figure 2. Catalog classification

Near circular low Earth, medium Earth, eccentric and low Earth orbits represent 95.6% of the orbit types belonging to the considered catalog. The values of the semi-major axis are between 1.018543 and 109.2196 Earth Radii in which 50% of the objects have $a < 1.5$ Earth Radii. The eccentricity is between 0.000001 and 0.9203, although almost 60% have $e < 0.01$. The inclination is between $0.00120°$ and $144.6415°$, including data of 270 objects near the critical inclination.

The TLEs were propagated over 7 days with DN, KB and the numerical method. The distance, along-track, cross-track, radial and relative errors of the orbital elements were calculated using both DN and KB methods.

We can now briefly summarize this analysis.

Fig. 3 shows the histogram of DN distance error. We should mention that the lack of robustness of estimates is due to the asymmetry on the right of the histogram. This asymmetry corresponds to TLEs with small eccentricity, $e < 0.01$, which are near the critical inclination.

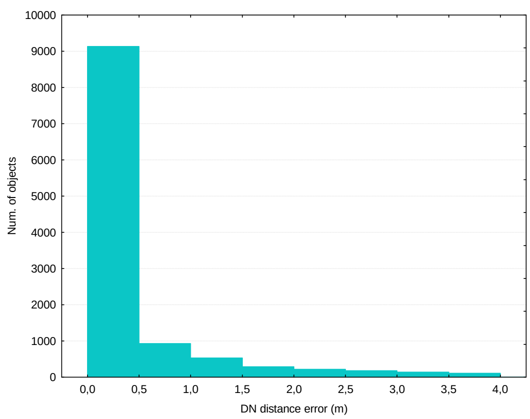

Figure 3. DN distance error histogram

Fig. 4 shows a box and whisker plot of the distance, along-track, cross-track and radial errors without upper outliers. These data have been classified in three sets: $e < 0.01$, $e \geq 0.01$ and $62.5° < i < 64.5°$, as well as other data. The most influential errors for TLEs with $e < 0.01$ were found in the along-track and radial components, whereas for TLEs with $e \geq 0.01$ and $62.5° < i < 64.5°$ the worst error behaviour was found in the cross-track component.

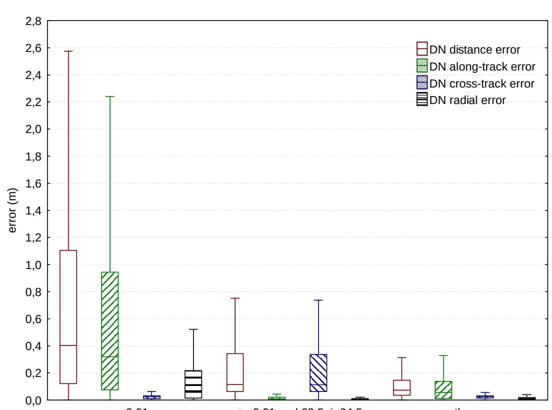

Figure 4. DN distance, along-track, cross-track and radial errors

The histogram of KB distance errors exhibits fairly symmetric behaviour, as can be seen in Fig. 5. We must bear in mind the robustness of estimates.

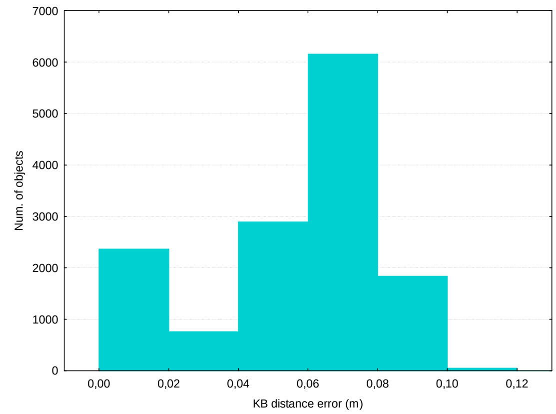

Figure 5. KB distance error histogram

Fig. 6 shows a box and whisker plot of the distance, along-track, cross-track and radial errors without upper outliers. These data have been classified as mentioned in the previous paragraph. However in this case the most influential errors for TLEs with $e < 0.01$ were found in the along-track and cross-track components, the worst error behaviour being found in the cross-track component, as in the previous DN case, for TLEs with $e \geq 0.01$ and $62.5° < i < 64.5°$. Note the scarce influence of radial error in the three cases.

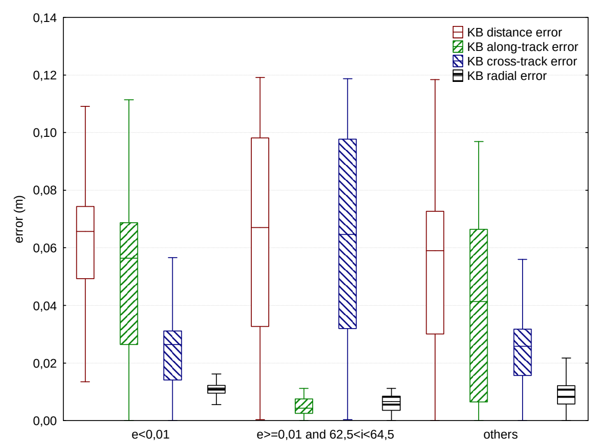

Figure 6. KrB distance, along-track, cross-track and radial errors

Fig. 7 and 8 shows a box and whisker plot analysis for DN and KB distance errors taking into account the classification suggested in[**?**]. Upper and lower outliers of less than 3.9 m are taken into account in this plot. It is worth noting that Fig. 8 shows lower outliers for LEO Circular category ($n > 4$, $a_p > 300$, $e \leq 0.05$) in which the

Krylov-Bogoliubov-Mitropolsky method provides good orbit determination compared with all other TLEs in this category.

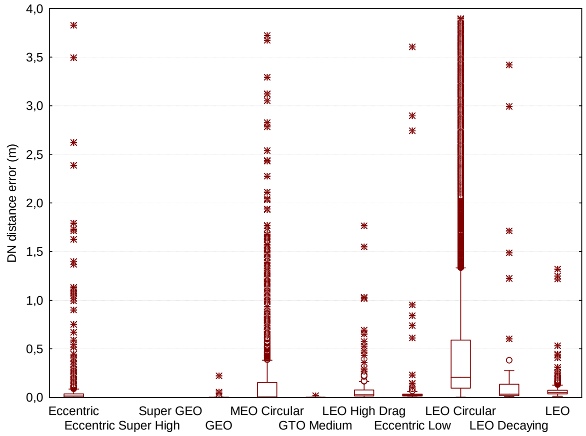

Figure 7. DN distance error histogram

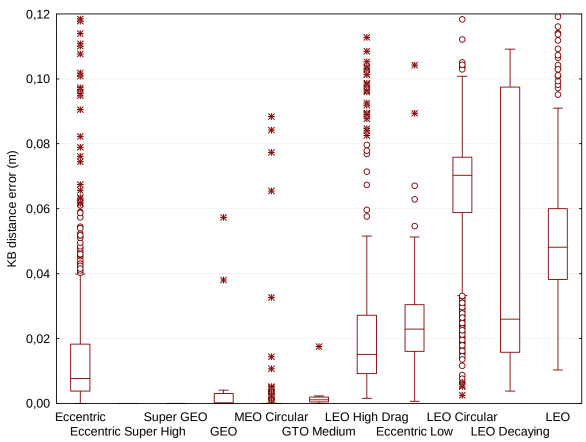

Figure 8. KB distance, along-track, cross-track and radial errors

5. CONCLUSIONS

The previous analysis over 7 days of propagation shows the robustness of estimates produced by the Krylov-Bogoliubov-Mitropolsky method vs. the Delaunay Normalization. In fact, the behaviour of the Krylov-Bogoliubov-Mitropolsky method is better than the Delaunay Normalization in 87% of all cases analyzed, in which the mean value is only 0.0083 m.

In particular, the worst behaviour of the Delaunay Normalization is found in the $e < 0.01$ category. It is worth noting that the relative semi-major axis error is only slightly better in DN case than in KB case, being 0.0035 m vs 0.0045 m in mean values respectively. Eccentricity, argument of the perigee and mean anomaly errors are much better in KB case than DN case , whereas the behaviour of the inclination and argument of the node errors are similar in both DN and KB.

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