# USE OF GENERALIZED FUNCTIONS FOR DEFINITION OF COLLISION INTEGRALS IN ORBITAL MOTION

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**ABSTRACT**

The estimation of probability of mutual collisions between orbital objects can be specified as a problem of calculation of collision integrals like it is in statistical physics. The approach based on a definition of the concentration (space density) of a single particle in a Keplerian frame as a generalized function is proposed to solve problems of this kind. The advantage of a such conception is that it makes the conversion from sky mechanics formulas to statistical problems more explicit.

## 1. INTRODUCTION

The problem of operational safety with respect to orbital collisions differs from the typical problems of the classical celestial mechanics through its statistical nature. In this case the mathematical problem brings to solution of integral or integro-differential equations for distribution function, and the formulas for orbital motion of an individual objects is of dependant character similar that the evaluation of slowing-down area of neutron is in reactor problems. The conversion from discrete conception on an orbital motion to statistical one can be realized using a generalized function approach. The idea of generalized function, on the one side, gives a possibility to express in mathematically proper form such an idealized representation as the space density of a material point. And on the other side, it is a reflection of the fact, that there is no possibility to measure a physical value, but some averaged value in a finite vicinity makes sense. So the generalized functions are a suitable method to describe distributions of different physical values, and they are also called as distributions. The generalized functions were inserted by P.A.M in his investigations on quantum mechanics at the end of twenties in the last century [Dirac P.A.M., 1932].
Let take up a one-dimensional function

$$\delta(x) = \begin{cases} 0 & \text{for all } x \text{ except } x = 0 \\ \infty & \text{if } x = 0 \end{cases}$$

It is imposed additionally

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1 .$$

It is not a usual function because in mathematics the symbol $\infty$ is not regarded as some defined value or a space point but as a consecution of points or the symbol of a limit. It can be said that the Dirac function is a generalization of the idea of matter density usually used in the continued approach for discontinuous objects, and it allows to change from a discrete description of the environment to a continued one and backwards.
Delta-function has an interesting behaviour which stipulates for using in integrals of convolution type – integrals by Stieltjes T. J. The Dirac operator transferring a function into other one has a view:

$$D(x-t) \otimes f(x) = \int_{-\infty}^{+\infty} \delta(x-t) f(x)dx ,$$

and $D(x-t) \otimes f(x) = f(t)$ .
The idea of delta-function can be used in three-dimensional space. Such definition for multi-dimensional function was given in the well known monograph by L. Landau on theoretical physics. The three-dimensional Cartesian space is considered, and the three-dimensional delta-function is a simple product of usual one-dimensional delta-functions:

$$\delta(x,y,z)=\delta(x)\delta(y)\delta(z)$$

The idea of a delta-function can be also used in a non-Euclidean space – a Keplerian space. Fidelity of the widening don't raise doubts because it is based on well known properties of the one-dimensional delta-function.
The properties and some effluent relations are following.
A separate particle that moves uniformly with velocity $v$, is described by a formula:

$$\rho(x) = \frac{1}{T}\int_{0}^{T} \delta(vt-x)dt$$

It is an averaged density of the distribution in 1-D space: the particle is overlapped over the interval $Tv$, which it moves through.
Here

$$\int_{-\infty}^{+\infty} \rho(x)dx = 1 .$$

The physical sense is: somewhere in space there is a particle.

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When a Keplerian motion is described reciprocal unambiguity of reflection of the usual Cartesian space on the Keplerian one must be considered. To understand the solution principles the problem of a discrete particle oscillating with frequency ω is considered:

$$\rho(x) = \frac{1}{T}\int_0^T \delta(\sin\omega t - x)dt = \frac{2}{T\omega}\int_{-1}^{+1} \delta(\xi - x)\frac{d\xi}{\sqrt{1-\xi^2}} .$$

Here $T$ – a period of oscillations, $T = \frac{2\pi}{\omega}$. If the time interval is equal of a some integer number of the oscillation periods $kT$, the integral repeats $k$ times, $k$ the denominator and the numerator are cancelled, and the same result is obtained. A notion: for the change of variables the polysemy character of the function $t = \frac{arcsin\,\xi}{\omega}$ must be considered, the polysemy leads to appropriate fragmentation of the integration region Also it must be considered that the particle passes every point twice for an oscillation. In the points of $x=\pm1$ the density turns to infinity.

In this case the space integral is equal unity:

$$\int_{-\infty}^{+\infty} \rho(x)dx = \int_{-1}^{+1} \frac{dx}{\pi\sqrt{1-x^2}} = 1$$

Notice that the same result can be used if the expression is integrated over the region where the sine is a mutually single-valued: a half-period [-$T$/2, +$T$/2].

$$\rho(x) = \frac{2}{T}\int_{-T/2}^{+T/2} \delta(\sin\omega t - x)dt$$

There is no ruses conditioned to the polysemy character of the functions used for the variable

A general case of 2D and 3D spaces: when a discrete particle moves in a 3D space the averaged over a time interval $T$ space is:

$$\rho(x,y,z) = \frac{1}{T}\int_0^T \delta(X(t)-x)\delta(Y(t)-y)\delta(Z(t)-z)dt .$$

Here the functions $X(t)$, $Y(t)$, $Z(t)$ are a parametric representation of the particle trajectory. This formula can be used also in derivation of equations for particles in a non-Euclidean space.

A distribution in geocentric coordinate frame when there is a motionless discrete particle: in this case the space distribution is

$$\rho_1(\varphi,\omega,r) = \delta(\sin\Phi - \sin\varphi)\cdot\delta(\Omega-\omega)\cdot\delta\left(\frac{R^3}{3}-\frac{r^3}{3}\right). \quad (1)$$

This delta-function is chosen having in view conditions of normalization and symmetry:

$$\int_0^\infty \delta\left(\frac{R^3}{3}-\frac{r^3}{3}\right)\cdot r^2 dr = 1 , \quad \int_{R_1}^{R_2} \delta\left(\frac{R^3}{3}-\frac{r^3}{3}\right)\cdot\frac{1}{V}\cdot R^2 dR = \frac{1}{V} .$$

The first equality shows that there is a particle in the space.

The second equality shows that the particle is equidistributed over the volume between radii $R_1$ and $R_2$.

## 2. ORBITAL MOTION

**A discrete particle moving in an orbit.**

Its coordinates vary in time as $\Phi = \Phi(t)$, $\Omega = \Omega(t)$ и $R = R(t)$. In this case the distribution depends on the time.

$$\rho_1(\varphi,\omega,r,t) = \delta(\sin\Phi(t) - \sin\varphi)\cdot\delta(\Omega(t)-\omega)\cdot\delta\left(\frac{R^3(t)}{3}-\frac{r^3}{3}\right)$$

Let the particle is moving in elliptical orbit with inclination $i$, longitude of ascending node $\Omega_0$ and argument of pericentre $\theta_0$. Without decreasing of commonality let $0 \le i \le \pi/2$. The distribution at any instant is $\rho_1(\varphi,\omega,r,t)$, that is obtained from (1) after substitution of the equations for Φ, Ω, and $R$ time dependent.

$$\rho_2(\varphi,\omega,r) = \frac{1}{T_1}\int_0^{T_1} \rho_1(\varphi,\omega,r,t)dt , \quad (2)$$

where $T_1 = \frac{2\pi}{\sqrt{(1-e^2)^3}}\sqrt{\frac{p^3}{\mu_0}}$ is an orbital period,

p is a focal parameter.

Using properties of delta-function δ(Ω-ω), for example, the following expression for the distribution is obtained (the particle is overlapped over its orbit):

:

$$\rho_2(\varphi,\omega,r) = \frac{r^2\sqrt{(1-e^2)^3}}{2\pi p^2}\delta(\sin\Phi - \sin\varphi)\delta\left(\frac{R^3}{3}-\frac{r^3}{3}\right)\cdot \cdot\frac{\cos i}{1-\sin^2 i\cos^2(\omega-\Omega_0)} \quad (3)$$

The values of Φ and $R$ are determined with the following expressions:

$$\sin\Phi = \frac{\sin i \sin(\omega-\Omega_0)}{\sqrt{1-\sin^2 i \cos^2(\omega-\Omega_0)}} \quad (4)$$

$$\cos\Phi = \frac{\cos i}{\sqrt{1-\sin^2 i \cos^2(\omega-\Omega_0)}} \quad (5)$$

$$R = \frac{p}{1+e\cos\theta} \quad (6)$$

$$\cos(\theta-\theta_0) = \frac{\cos i \cos(\omega-\Omega_0)}{\sqrt{1-\sin^2 i \cos^2(\omega-\Omega_0)}} \quad (7)$$

**A precessing orbit case.**

Let in the equation (3) the longitude of ascending node $\Omega_0$ is precessing with a period $T_2$ Then in the equations (3)-(7) the time dependence $\Omega_0(t)$ occurs, and this correlation determines the time dependant function $\rho_2(\varphi,\omega,R,t)$. Averaging over the period $T_2$, the density is obtained in the case when the particle is overlapped over the surface of a rotation ellipsoid:

$$\rho_3(\varphi,\omega,r) = \frac{1}{T_2}\int_0^{T_2} \rho_2(\varphi,\omega,r,t)dt\,, \quad (8)$$

Using properties of delta-function $\delta(\sin\Phi - \sin\varphi)$, and considering that for the whole period of the precession every value $\sin\Phi$ repeats twice, and considering the relations between the precession period and the speed of motion of the longitude of ascending node $T_2 = \frac{2\pi}{\frac{d\Omega_0}{dt}}$, the following equation is obtained:

$$\rho_3(\varphi,\omega,r) = \frac{r^2\sqrt{(1-e^2)^3}}{2\pi^2 p^2}\left[\delta(\frac{R_1^3}{3}-\frac{r^3}{3})+\delta(\frac{R_2^3}{3}-\frac{r^3}{3})\right]\cdot \cdot\frac{1}{\sqrt{\sin^2 i-\sin^2\varphi}} \quad (9)$$

Here $-i \le \varphi \le i$,

$$R_1 = \frac{p}{1+e\cos\theta}\,, \qquad R_2 = \frac{p}{1-e\cos\theta}\,,$$

$$\cos(\theta-\theta_0) = \frac{\sqrt{\sin^2 i-\sin^2\varphi}}{\sin i}\,.$$

Now consider that in the equation (9) the argument of a perigee $\theta_0$ precesses with period of $T_3 = \frac{2\pi}{\frac{d\theta_0}{dt}}$. Such phenomenon leads to overlapping of the particle over some body like a ring: $r_1 \le r \le r_2$ , $-i \le \varphi \le i$. Here $r_1$ and $r_2$ - a perigee and an apogee of the orbit. Averaged density is:

$$\rho_4(\varphi,\omega,r) = \frac{1}{T_3}\int_0^{T_3}\rho_3(\varphi,\omega,r,t)dt\,,$$

Considering that the particle passes through every value of radius twice an orbit, the following expression is obtained:

$$\rho_4(\varphi,\omega,r) = \frac{1}{2\pi^3}\frac{1}{\sqrt{\sin^2 i-\sin^2\varphi}}\frac{(1-e^2)^{3/2}}{pr^2\sqrt{e^2-\left(\frac{p}{r}-1\right)^2}}\,, \quad (10)$$

where $-i \le \varphi \le i$, $r_1 \le r \le r_2$..

The equation (9) can be written in a symmetric shape by D. Kessler as following: :

$$\rho_4(\varphi,\omega,r) = \frac{1}{2\pi^3 ar}\frac{1}{\sqrt{(\sin^2 i-\sin^2\varphi)}\cdot\sqrt{(r-r_1)(r_2-r)}}\,,$$

where $a = (r_1 + r_2)/2$ – a major half-axis of an orbit. Notice that all above distributions (1),(3),(8), and (9) satisfy the normalization condition:

$$\int_0^{2\pi} d\omega \int_{-\pi/2}^{\pi/2} \cos\varphi\cdot d\varphi \int_0^{\infty} r^2 dr \times \rho(\varphi,\omega,r) = 1$$

All this equations can be used for orbits with $i > \pi/2$, if $i$ substitute for $\pi - i$.

## CONCLUSION

Thus the formal approach to development of averaged space density of orbital objects are proposed. This approach is not so simple but can be used due its generality. This technique allows to get explicit equations for different statistical problems of celestial mechanics. In particular, there is obtained the expression for the collisional integral before derived in [Opik E.J., 1951], [Wetherwlll G.W., 1967], [Kessler D J., 1981], and [Nazarenko A.I.., 1993] .

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