EFFICIENT ESTIMATION AND DECISION-MAKING METHODS FOR SHORT TRACK IDENTIFICATION AND ORBIT DETERMINATION

A. G. Tartakovsky, A. E. Kolessa, A. P. Ivanov, A. P. Lukyanov, and V. A. Radchenko

Space Informatics Laboratory, Moscow Institute of Physics and Technology, Moscow, Russia, Email: agt@phystech.edu; kolessa.ae@phystech.edu; atner.ivanov@phystech.edu; lukianov.ap@mipt.ru; rad4enko.v.a@mail.ru

ABSTRACT

Detection and tracking of near-Earth space objects with telescopes involve sparse data, i.e., the observations related to a specific object of interest are collected only for a small fraction of the object's orbit revolution, which is referred to as a short arc. The time-separation between a series of successive observations of the same object can be very large (a week or even more). The challenge is to estimate orbit parameters for a single track and then to improve this estimate based on a few data sets, say, fusing two tracks. Our paper addresses the problem of identification and fusion of several short optical tracks of near-Earth space objects, as well as the problem of estimation of parameters of the corresponding orbit directly from these tracks in the absence of *a priori* information on the object's orbit. The popular approach to solving these problems is based on admissible regions of orbital parameters, which is typically computationally demanding. A new, purely statistical method for solving these problems is proposed. This method includes a novel algorithm of estimation of orbit parameters based on two time-separated tracks and an optimal joint track identification-estimation-fusion rule, i.e., the decision-making rule on whether the tracks belong to the same object or not that also simultaneously estimates parameters. An important feature of the developed algorithm is that it allows one to find the global minimum of the objective function instead of localizing local minima.

Keywords: Admissible Regions, Detection and Tracking, Iterative Weighted Least Squares Method, Joint Hypothesis Testing and Estimation, Orbit Determination, Short-Arc Tracks Identification.

1. INTRODUCTION

The paper addresses the problem of identification and fusion of several short optical tracks of near-Earth non-maneuvering space objects, as well as the problem of estimation of parameters of the corresponding orbit directly from these tracks in the absence of *a priori* information on the object's orbit. A popular approach to these problems is based on the regions of admissible values of the orbital parameters (see, e.g., [6, 8–10, 15, 18]). The predicted admissible region for the first track is compared to the admissible region for the second track, and if they intersect, the decision is made that the tracks belong to the same space object, and the region of their intersection is used for refining the estimate. However, if the time interval between tracks is long, admissible regions have the form of extended ravines of complex shape with transverse dimensions many orders of magnitude smaller than the longitudinal dimensions, which makes it difficult to obtain a reliable and accurate estimate. Also, this approach is computationally demanding.

In the present paper, we exploit a completely different, purely statistical approach that does not require admissible regions of orbital parameters. This approach is based on the two novel methods developed by Kolessa et al. [13]. The first is a decision-theoretic approach to the problem of joint identification-estimation-fusion of tracks that allows one to obtain an optimal Bayesian hypothesis test for testing the hypothesis that tracks belong to different objects against the hypothesis that tracks belong to a single object. If the hypothesis test makes a decision that the tracks belong to the same object they are fused and the refined estimate of the object's parameters is formed based on the novel nonlinear estimation algorithm, also developed in this paper. To build the latter algorithm the results obtained in [12] for a single track are extended to the case of two short tracks, showing that the weighted least-squares criterion for finding the optimal estimate of the orbital parameters based on two short tracks has several extremes. Thus, to solve the estimation problem correctly it is important to localize the global minimum rather than local ones. For completeness, we provide a detailed description of all ideas and algorithms developed in [12, 13].

The results of experiments with real data show very high efficiency of the developed methods. In fact, the experimental study proves that the accuracy of estimation of the orbit based on two or three time-separated tracks is close to potentially achievable.

Proc. 1st NEO and Debris Detection Conference, Darmstadt, Germany, 22-24 January 2019, published by the ESA Space Safety Programme Office
Ed. T. Flohrer, R. Jehn, F. Schmitz (http://neo-sst-conference.sdo.esoc.esa.int, January 2019)

2. ALGORITHMS OF ORBIT DETERMINATION

2.1. A Single Track Problem

We begin with describing an algorithm of estimation of a space object orbit based on angle-only measurements in one observation session, i.e., from a single track. This problem was considered in a number of works. The first results were obtained by Legendre [14] and Gauss [11]. Chang [4] proposed an algorithm based on independent smoothing of the measurements of each of the two angular coordinates by a polynomial of the appropriate degree. Estimates of the unknown range and radial velocity are found as the roots of the transcendental equation connecting these parameters with the angular coordinates and their derivatives. Smoothing the measured angular coordinates with polynomials (but not based on the differential equation of motion) leads to a loss of potential accuracy and, in a number of cases, the transcendental equation has no real roots, which makes it impossible to obtain an estimate of the orbital parameters. In [4], this method was used to form the initial approximation and then refine it by an iterative least-squares method. The analysis in [4] shows that in the general case it is not possible to ensure a stable convergence of such an iterative process. In this subsection, we describe the method for the initial determination of orbits, which refines the algorithm proposed in [12].

Introduce the following notation that will be used throughout the paper. A track of the space object is the sequence

$$\left(\hat{\alpha}_t, \hat{\delta}_t, \tau_t\right), \quad t = 0, 1, \ldots, n-1, \tag{1}$$

where $\hat{\alpha}_t$ and $\hat{\delta}_t$ are the measurements of the angle coordinates $\alpha(\tau_t)$ (right ascension) and $\delta(\tau_t)$ (declination), respectively, corresponding to the time moment τ_t. The range $d(\tau_t)$ from a telescope to an object is not measured. Write $X(\tau^*)$ for the state vector of the space object equation of motion corresponding to some point in time τ^*. This vector contains k components, which correspond to vectors of location and velocity of the object in the geocentric fixed Cartesian system TEME (TrueEquatorMeanEquinox). If necessary, the state vector may also include components that characterize the space object inhibition in the atmosphere and the effect of solar radiation on its motion. The law of the space object motion is determined by the equation of the prediction of the state vector to an arbitrary time τ:

$$X(\tau) = f(X, \tau - \tau^*). \tag{2}$$

Measured angular coordinates are functions of the state vector X:

$$\alpha(\tau_t) = \alpha(X(\tau_t)) = \alpha(f(X, \tau_t - \tau^*)), \tag{3}$$

$$\delta(\tau_t) = \delta(X(\tau_t)) = \delta(f(X, \tau_t - \tau^*)), \tag{4}$$

$t = 0, 1, \ldots$. The following model for the measurements of the angular coordinates (observations) is used:

$$Y(t) = h(X, \tau_t) + \varepsilon_t, \tag{5}$$

where

$$h(X, \tau_t) = \left[\alpha(f(X, \tau_t - \tau^*)), \delta(f(X, \tau_t - \tau^*))\right]^T, \\ \varepsilon_t = (\varepsilon_t^\alpha, \varepsilon_t^\delta)^T, \quad \text{for } t = 0, 1, \ldots, n-1, \tag{6}$$

ε_t^α and ε_t^δ are measurement errors, Y is the observed random variable, and its observed (measured) value is denoted by $y(t) = (\hat{\alpha}_t, \hat{\delta}_t)^T$. Hereafter T denotes transpose. Using the notation

$$\begin{aligned} Y &= \left[Y^T(0), \ldots, Y^T(n-1)\right]^T, \\ y &= \left[y^T(0), \ldots, y^T(n-1)\right]^T, \\ \varepsilon &= \left[\varepsilon^T(0), \ldots, \varepsilon^T(n-1)\right]^T, \\ h(X) &= \left[h^T(X, \tau_0), \ldots, h^T(X, \tau_{n-1})\right]^T \end{aligned} \tag{7}$$

the combined vector of measurements can be represented in the vector form

$$Y = h(X) + \varepsilon. \tag{8}$$

Assume that the measurement error ε has a normal distribution with mean zero and diagonal covariance matrix $W = \sigma^2 \mathbb{1}$, where the variance σ^2 of the average error in the measurement of angular coordinates is unknown ($\mathbb{1}$ stands for the unit matrix). We impose the following constraint on the admissible values of the state vector of the space object motion:

$$X \in \Omega \subset \mathbb{E}_k, \tag{9}$$

where $\mathbb{E}_k$ stands for the k-dimensional Euclidean space.

Based on the observed value y of the random vector Y, it is required to estimate $\hat{x}$ of the orbit state vector X, which is optimal in the sense of the Gauss–Markov criterion:

$$J(\hat{x}) = \min_{x \in \Omega} J(x), \tag{10}$$

where

$$J(x) = [y - h(x)]^T [y - h(x)], \tag{11}$$

as well as to find the posterior covariance matrix of the estimation error $\Gamma = \mathsf{E}[(X - \hat{x})(X - \hat{x})^T | Y = y]$.

Unfortunately, in the case of a short track, it is impossible to construct an initial approximation for the iterative least squares method, ensuring its convergence to $\hat{x}$ that delivers the minimum of the optimality criterion (11). This is due to the essential "ravineness" of the objective function (11). In addition, the application of an iterative least squares method with one initial point makes it difficult to take into account the *a priori* constraint (9) on the admissible values of the state vector, which is necessary to construct a proper confidence region.

Consider first fixed range orbit estimation, that is, the problem (8)–(11) for a fixed hypothetical range d_0 from the telescope to the object with an additional constraint

$$\alpha^T X = d_0 + \hat{e}_0^T x_s, \tag{12}$$

where $\alpha^T = (\hat{e}_0^T, 0, 0, 0)$, $\hat{e}_0 = e(\hat{\alpha}_0, \hat{\delta}_0)$ is a unit vector, directed from the telescope to the object, and x_s are the coordinates of the telescope. The goal is to find an estimate $\hat{x}(d_0)$ of the state X for a given range d_0.

Fixing the hypothetical range d_0 from the telescope to the object and using [12], we first obtain the initial conditional estimate $\tilde{x}(y|d_0)$ of the state vector X. Then linearizing equation (8) with respect to the estimate $\tilde{x}(y|d_0)$ and applying the least squares method with the restriction (12) gives the following estimator:

$$\hat{x}(d_0) = \hat{x}^{\text{LSE}} + \hat{\Gamma}^{\text{LSE}} \frac{\alpha \left(\alpha^T \hat{x}^{\text{LSE}} - a\right)}{\alpha^T \hat{\Gamma}^{\text{LSE}} \alpha},$$

where

$$\hat{\Gamma}^{\text{LSE}} = \left[\left(\frac{\partial h(\tilde{x}(y|d_0))}{\partial x}\right)^T \left(\frac{\partial h(\tilde{x}(y|d_0))}{\partial x}\right)\right]^{-1},$$

$$\hat{x}^{\text{LSE}} = \tilde{x}(y|d_0) + \hat{\Gamma}^{\text{LSE}} \left(\frac{\partial h(\tilde{x}(y|d_0))}{\partial x}\right)^T (y - h(\tilde{x}(y|d_0)))$$

and $a = d_0 + \hat{e}_0^T x_s$ (cf. [1]).

Finally, consider the most practical case when the range is not known but belongs to some interval, which can be calculated based on physical and tactical conditions. The function $\hat{x}(d_0)$ allows us to determine such interval $(d_{\min}, d_{\max})$ that $\hat{x}(d_0) \in \Omega$. This makes it possible to reduce the nonlinear problem of minimizing the objective function (11) in a multidimensional space under constraints $x \in \Omega$ to the minimization problem in one-dimensional space:

$$J(\hat{x}(\hat{d}_0)) = \min_{d_0 \in (d_{\min}, d_{\max})} J(\hat{x}(d_0)). \tag{13}$$

The solution of this minimization problem can be obtained, for example, by Brent's numerical method [2]. The estimate $\hat{x}(\hat{d}_0)$ is refined by the iterative least squares method

$$\hat{\Gamma}_{i+1} = \left[\left(\frac{\partial h(\hat{x}_i)}{\partial x}\right)^T \left(\frac{\partial h(\hat{x}_i)}{\partial x}\right)\right]^{-1},$$

$$\hat{x}_{i+1} = \hat{x}_i + \hat{\Gamma}_i \left(\frac{\partial h(\hat{x}_i)}{\partial x}\right)^T (y - h(\hat{x}_i)).$$

Iterations are performed with the initial condition $\hat{x}_0 = \hat{x}(\hat{d}_0)$ and terminated when the condition

$$|J(\hat{x}_{i+1}) - J(\hat{x}_i)| \leqslant \delta \quad \text{or } i > i_{\max} \tag{14}$$

is satisfied, where the constant δ determines the accuracy of the result and $i_{\max}$ is a maximal admissible number of iterations.

The value of $\hat{x}_{i+1}$ (obtained when condition (14) is satisfied) is a solution to the problem (10)–(11), i.e., $\hat{x}(d_0) = \hat{x}_{i+1}$, and the matrix $\hat{\Gamma} = \hat{\Gamma}_{i+1}$ is the covariance estimation error matrix normalized to an unknown variance σ^2 of measurement error. Notice that both the estimate $\hat{x}$ and the normalized matrix $\hat{\Gamma}$ do not depend on the unknown σ.

The average error (over the track) σ^2 in the measurement of angular coordinates is estimated as

$$\hat{\sigma}^2 = \frac{J(\hat{x})}{2n - k} \tag{15}$$

and the covariance matrix of the estimation error has the form $\Gamma = \hat{\sigma}^2 \hat{\Gamma}$.

Obviously, J/σ^2 has the standard chi-squared distribution with $2n - k$ degrees of freedom, χ^2_{2n-k}. Hence, the confidence region (ellipsoid) containing the state vector X with required significance level β is given by the inequality

$$Ell(\hat{x}, \Gamma|\beta) = \{X : (X - \hat{x})^T \Gamma^{-1} (X - \hat{x}) \leqslant c^2\}, \tag{16}$$

where the constant $c = c_\beta$, depending on the prescribed significance probability β, is found from the equation

$$F_{k,2n-k}\left(\frac{c^2}{2n - k}\right) = \beta, \tag{17}$$

where $F_{m,N}(x)$ is the Fisher distribution function,

$$F_{m,N}(x) = \frac{\Gamma\left(\frac{N+m}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_0^x \frac{u^{m/2-1}}{(1+u)^{(N+m)/2}} \, \mathrm{d}u.$$

Taking into account the *a priori* constraint (9), the confidence region $\widehat{\Omega}$ for the state vector has the form:

$$\widehat{\Omega} = \Omega \bigcap Ell(\hat{x}, \Gamma|\beta). \tag{18}$$

The developed single-track estimation algorithm was tested based on the tracks of the objects of Space Station, Atlas 5 Centaur R/B, Beidou 3M8, observed from Moscow, Russia with a telescope with angular accuracy $1''$ and a measurement acquisition period of 5 sec. The parameters of the orbits of these objects are given in Table 1, where the ID is the NORAD catalog number, P is the perigee, A is the apogee, I is the inclination, and T is the period. In Tables 2–4, we give dependence on the time of observation of the root-mean-squared estimation errors (RMSE) of position σ_{pos} and velocity σ_{vel} of the object as well as the number of turns of the virtual object's spiral.

Table 1. Analyzed Orbits

No.	ID	P (km)	A (km)	I	T (min)
1	25544	409	415	51.6^o	92.7
2	42916	4520	34816	25.9^o	697
3	43246	21520	21550	55.0^o	773

Table 2. Object 25544

T (sec)	$\sigma_{\rm pos}$ (km)	$\sigma_{\rm vel}$ (m/sec)	N
20	13.67	84.85	2.571
30	5.50	34.41	1.583
60	1.07	6.86	0.368
120	0.19	1.22	0.066
200	0.05	0.35	0.019
300	0.02	0.15	0.008

Table 3. Object 42916

T (sec)	$\sigma_{\rm pos}$ (km)	$\sigma_{\rm vel}$ (m/sec)	N
200	114.6	38.2	0.989
300	48.8	16.1	0.415
600	9.5	3.0	0.080
900	3.9	1.2	0.032
1200	2.1	0.6	0.016
2000	0.7	0.2	0.005

Table 4. Object with Norad Number 43246

T (sec)	$\sigma_{\rm pos}$ (km)	$\sigma_{\rm vel}$ (m/sec)	N
300	1060.8	159.93	2.457
600	121.3	18.24	0.420
900	45.7	6.84	0.155
1200	21.3	3.18	0.072
2000	5.3	0.79	0.018
3000	1.6	0.24	0.005

2.2. Estimation of Orbit Parameters Based on Two Time-Separated Tracks

2.2.1. The Problem

Consider two tracks of the same object

$$(\hat\alpha_{0,t}, \hat\delta_{0,t}, \tau_{0,t}), \quad t = 0, 1, \ldots, n_0 - 1,$$
$$(\hat\alpha_{1,t}, \hat\delta_{1,t}, \tau_{1,t}), \quad t = 0, 1, \ldots, n_1 - 1,$$

obtained by one or different telescopes at one or different orbital turns, possibly on different observation nights. For the sake of simplicity, the tracks are numbered in descending order of their duration in time.

Using notations (1)–(8), we consider the state vector $X_0 = X(\tau_{0,0})$ to define the motion of the object and the following equations for the combined measurable vectors

$$Y_0 = h_0(X_0) + \varepsilon_0, \;\; Y_1 = \tilde h_1(X_0) + \varepsilon_1, \qquad (19)$$

where $\tilde h_1(X_0) = h_1(f(X_0, \tau_{1,0} - \tau_{0,0}))$, Gaussian measurement errors ε_0 and ε_1 are uncorrelated, and their covariance matrices have the diagonal form $W_0 = \sigma_0^2 \mathbb{1}$ and $W_1 = \sigma_1^2 \mathbb{1}$. Variances σ_0^2 and σ_1^2 of the average (over tracks) errors of measurements of angular coordinates are unknown.

Introduce notation:

$$y = (y_0, y_1)^T, \quad Y = (Y_0, Y_1)^T,$$
$$H_0(X_0) = [h_0(X_0), \tilde h_1(X_0)]^T,$$
$$Y = H_0(X_0) + \varepsilon, \;\; W = \begin{bmatrix} \sigma_0^2 \mathbb{1} & 0 \\ 0 & \sigma_1^2 \mathbb{1} \end{bmatrix}. \qquad (20)$$

Based on the measurement y of the random vector Y, we have to obtain the *fused* estimate $\hat x_0$ of the orbit state vector X_0, which is optimal in the sense of the Gauss–Markov criterion:

$$J(\hat x_0) = \min_{x_0 \in \Omega} J(x_0), \qquad (21)$$

where

$$J(x_0) = [y - H_0(x_0)]^T \widehat{W}^{-1} [y - H_0(x_0)], \qquad (22)$$
$$\widehat{W} = \begin{bmatrix} \hat\sigma_0^2 \mathbb{1} & 0 \\ 0 & \hat\sigma_1^2 \mathbb{1} \end{bmatrix},$$

and $\hat\sigma_0^2$, $\hat\sigma_1^2$ are the estimates of the average variances of error measurements constructed for each track independently according to the formula (15). It is also required to determine the posterior covariance matrix of the state vector error estimates.

While at first glance the problem of minimizing the criterion (21) is the same as the problem of estimating the state vector based on a single track (i.e., finding the global minimum of criterion (10)), this is not correct. Indeed, as discussed in [13], there is a fundamental difference due to a long time interval between the measurements of the first and second tracks and a long-term prediction $f(X_0, \tau_{1,0} - \tau_{0,0})$ necessary for describing the connection of the second track with the state vector X_0. The details are spelled out in the next subsection.

2.2.2. An Optimal Estimate

Suppose that for the first track, i.e., based on the measurement y_0, we calculated an estimate $\tilde x_0$ of the state vector X_0 and a covariance matrix of estimation errors $\tilde\Gamma_0$ that determine the confidence ellipsoid $Ell(\tilde x_0, \tilde\Gamma_0|\beta)$ covering the state vector X_0 with the desired significance probability β. Let us represent the matrix $\tilde\Gamma_0$ in the block form

$$\tilde\Gamma_0 = \begin{bmatrix} \Gamma_{x,x} & \Gamma_{x,\nu} \\ \Gamma_{x,\nu}^T & \Gamma_{\nu,\nu} \end{bmatrix},$$

where each block has the size 3×3. Components with indices x correspond to object location and components

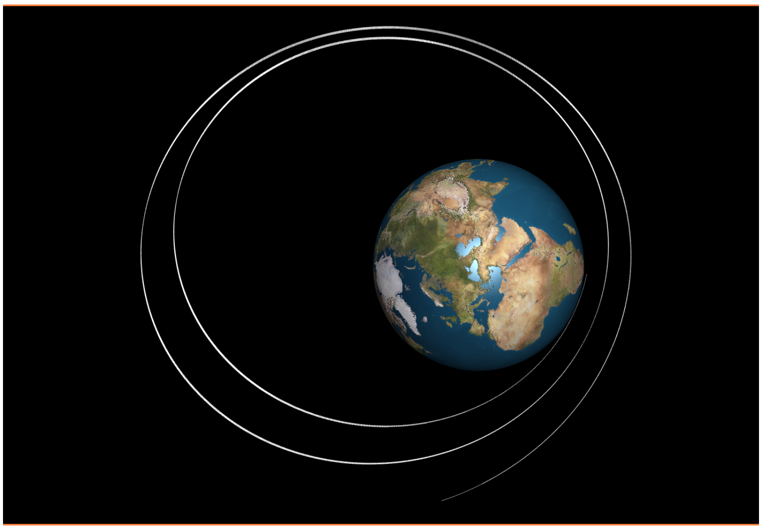

Figure 1. Spiral of the virtual object $x_0(u)$.

with indices ν correspond to velocity. Transforming the blocks $\Gamma_{x,x}$ and $\Gamma_{\nu,\nu}$ to the diagonal form by using orthogonal matrices S_x and S_ν,

$$\Lambda_x = S_x^T \Gamma_{x,x} S_x, \quad \Lambda_\nu = S_\nu^T \Gamma_{\nu,\nu} S_\nu.$$

and using the orthogonal matrix

$$S_1 = \begin{bmatrix} S_x & 0 \\ 0 & S_\nu \end{bmatrix}$$

the matrix $\tilde{\Gamma}_0$ can be written as

$$\Gamma_0' = S_1^T \tilde{\Gamma}_0 S_1 = \begin{bmatrix} \Lambda_x & S_x^T \Gamma_{x,\nu} S_y \\ S_y^T \Gamma_{x,\nu}^T & \Lambda_\nu \end{bmatrix}.$$

Suppose that at the position i, i of the matrix $\Gamma_0^{'}$ there is a maximal diagonal element $\Lambda_{i,i}^x$, corresponding to the coordinate space, and at the position $j+3, j+3$ – the maximal diagonal element $\Lambda_{j,j}^\nu$. There exists an orthogonal matrix S_2, which by means of the transformation

$$\Gamma_0'' = S_2^T \tilde{\Gamma}_0 S_2 = S_2^T S_1^T \tilde{\Gamma}_0 S_1 S_2 = S^T \tilde{\Gamma}_0 S$$

represents the matrix Γ_0' in such a form that (2×2)-submatrix of Γ_0'', formed by elements located at the intersection of rows and columns with numbers i and $j+3$, becomes orthogonal. In this case, the components i and $j+3$ of the vector $X_0'' = S^T X_0$, which variations give the greatest contribution to the change in the state vector X_0 when it is extrapolated, become independent.

The vector X_0'' has normal distribution with mean $\hat{x}_0'' = S^T \hat{x}_0$ and covariance matrix $\Gamma_0'' = S \tilde{\Gamma}_0 S^T$. The confidence ellipsoid corresponding to the matrix $\Gamma_0^{''}$ in the space of locations is most elongated along the axis i.

Let the vector $x_0''(u)$ be formed from the vector $\hat{x}_0''$ by adding to its i-th component the parameter u. The point $x_0''(u)$ corresponds to a k-dimensional point $x_0(u) = S x_0''(u)$ in the ellipsoid $Ell(\tilde{x}_0, \tilde{\Gamma}_0 | \beta)$.

Now, we find the interval $u_{\min} \leqslant u \leqslant u_{\max}$ of the admissible values of the parameter u, within which, firstly, the i-th component of the vector $x_0''(u)$ belongs to the confidence interval

$$\left(\hat{x}_0^{\prime\prime(i)} - \gamma \sqrt{\Gamma_0^{\prime\prime(i,i)}}, \hat{x}_0^{\prime\prime(i)} + \gamma \sqrt{\Gamma_0^{\prime\prime(i,i)}} \right)$$

with the desired probability β and, secondly, the vector $x_0(u)$ corresponds to the physically realizable orbit.

For an orbit given by the vector $x_0(u)$, one can determine the revolution period $T(u) = T(x_0(u))$ and, in particular, find $T_1 = T(u_{\min})$ and $T_2 = T(u_{\max})$.

A virtual object with the state vector $x_0(u)$ with a prediction for an interval of time $\Delta\tau = \tau - \tau_{0,0}$ will make $N(u) = \Delta\tau / T(u)$ revolutions around the Earth (the number $N(u)$ is not an integer). When the parameter u changes within the interval $[u_{\min}, u_{\max}]$, the state vector $x^E(u, \Delta\tau) = f(x_0(u), \Delta\tau)$ moves in a coordinate space along a spiral (see Figure 1) with the number of revolutions $N = |N_1 - N_2|$, where $N_1 = \Delta\tau / T_1$ and $N_2 = \Delta\tau / T_2$.

Using formulas (3) and (4), we can calculate the corresponding angular coordinates $\alpha(u, \tau_{1,t}) = \alpha(x_t^E(u))$, $\delta(u, \tau_{1,t}) = \delta(x_t^E(u))$, where

$$x_t^E(u) = f(x(u), \tau_{1,t} - \tau_{0,0}).$$

Angular discrepancy

$$\rho_{1,t}^2(u) = [\alpha(u, \tau_{1,t}) - \hat{\alpha}_{1,t}]^2 + [\delta(u, \tau_{1,t}) - \hat{\delta}_{1,t}]^2$$

as a function of the parameter u, periodically (with a spiral period) takes the minimum values since the point $x_t^E(u)$, moving in a spiral as the parameter u changes, periodically approaches as close as possible to the bearing beam, determined by the angular measurement $(\hat{\alpha}_{1,t}, \hat{\delta}_{1,t})$. The minimum value of the residual $\rho^2_{1,t}(u)$ is different for different turns of the spiral. The total quadratic residual for the second track has the same dependence of change on u:

$$\rho^2(u) = \sum_{t=0}^{n_1-1} \rho^2_{1,t}(u).$$

Simulations show that $1/T(u) \approx \eta \cdot u$, where η is a certain proportionality constant. This approximate formula allows us to calculate the period $T(u)$ of the function $\rho^2(u)$. Once we know the period $T(u)$, it is possible to determine the intervals

$$(u^{(k)}_{\min}, u^{(k)}_{\max}), \quad k = 0, 1, \dots, K-1$$

where the corresponding local minimum of the quadratic discrepancy for the second track is located. However, the discrepancy for the first track remains practically unchanged, since the point $x_0(u)$ is located in the confidence ellipsoid $Ell(\tilde{x}_0, \tilde{\Gamma}_0|\beta)$.

Thus, for a long time-interval between tracks, the quality criterion (21) has several extremes, the number of which increases with the increase of the interval between tracks. As a result, it is incorrect to use an iterative least-squares method with a single initial approximation for constructing an optimal estimate, which at best can converge to one of the local minima of the criterion (22). Hence, the problem of finding a global minimum of the criterion (22) becomes challenging. It is considered in the next subsection.

2.2.3. Finding the Global Minimum

Taking into account the analyzed features of the optimality criterion (22) for finding its global minimum, the following algorithm is proposed in each interval $(u^{(k)}_{\min}, u^{(k)}_{\max})$.

First, by using the method of mathematical optimization BRENT we find the value $u = \hat{u}_k$ providing a local minimum of the residual

$$\hat{\rho}_k = \rho(\hat{u}_k) = \min_{u^k_{\min} \leqslant u \leqslant u^k_{\max}} \rho(u) \tag{23}$$

as well as the corresponding state vector $x^{(k)} = x(\hat{u}_k)$.

Second, we determine a locally optimal estimate $\hat{x}^{(k)}$ (within the interval $(u^{(k)}_{\min}, u^{(k)}_{\max})$ under consideration) using the Kalman iterative filter:

$$\Gamma_i^{(k)} = \left(\tilde{\Gamma}_0^{-1} + B_i^{(k)T} \widehat{W}^{-1} B_i^{(k)}\right)^{-1}. \tag{24}$$

$$\hat{x}^{(k)}_{i+1} = \hat{x}^{(k)}_i + \Gamma^k_i \left[B_i^{(k)T} \widehat{W}^{-1} \left(y - h(\hat{x}^{(k)}_i)\right) + \tilde{\Gamma}_0^{-1}\left(\tilde{x}_0 - \hat{x}^{(k)}_i\right)\right], \tag{25}$$

where $B_i^{(k)} = \partial H_0(\hat{x}_i^{(k)})/\partial x$, with the initial condition $\hat{x}_0^{(k)} = x^{(k)}$. Iterations are terminated when the condition

$$\left(\hat{x}^{(k)}_{i+1} - \hat{x}^{(k)}_i\right)^T \left(\hat{x}^{(k)}_{i+1} - \hat{x}^{(k)}_i\right) \leqslant \delta$$

is satisfied, where δ is the prescribed estimation accuracy. The values of $\hat{x}^{(k)} = \hat{x}^{k)}_{i+1}$ and $\Gamma^{(k)} = \Gamma^{(k)}_{i+1}$ are taken as a state estimate and a covariance matrix of estimation errors, respectively.

Third, we calculate the local minimum of the objective function (22), $J^{(k)} = J(\hat{x}^{(k)})$.

After repeating the described sequence of operations for $k = 0, 1, \dots, K-1$, the number of the revolution $k^* = \arg\min_k J^{(k)}$ is determined, and the estimate $\hat{x}_0 = \hat{x}^{(k^*)}$ and the covariance matrix $\Gamma_0 = \Gamma^{(k^*)}$ are selected as the optimal estimate of the state vector X_0 and the covariance matrix of the estimation error constructed from two tracks in terms of criterion (21).

3. IDENTIFICATION AND FUSION OF TRACKS FROM DIFFERENT OBSERVATION SESSIONS

It is natural to formulate the problem of identification of tracks as belonging to the same or different objects as the problem of testing the hypothesis H_0 that both tracks belong to the same object whose orbit is determined by the state vector X_0 corresponding to the time instant $\tau_{0,0}$ versus the alternative hypothesis H_1 that two tracks refer to different objects whose orbits are determined by the state vectors $X_{1,0}$ and $X_{1,1}$ corresponding to the time instants $\tau_{0.0}$ and $\tau_{1,0}$.

Under these hypotheses the models for the observations are different. Specifically, under the hypothesis H_0, the observation model has the form (20), i.e.,

$$Y = H_0(X_0) + \varepsilon,$$

and under the hypothesis H_1, the observation model has the form

$$Y = H_1(X_1) + \varepsilon, \tag{26}$$

where

$$Y = (Y_0^T, Y_1^T)^T, \quad X_1 = (X_{1,0}^T, X_{1,1}^T)^T,$$
$$H_1(X_1) = (h_0^T(X_{1,0}), h_1^T(X_{1,1}))^T, \quad \varepsilon = (\varepsilon_0^T, \varepsilon_1^T)^T.$$

Observing the value y of the random vector Y we have to make a decision on which of the hypotheses is true and construct the estimate $x_0^*(y)$ of the vector X_0 when accepting the hypothesis H_0 or the estimate $x_1^*(y)$ of the combined state vector X_1 when accepting the hypothesis H_1.

For developing an optimal track identification algorithm we consider a Bayesian joint hypothesis testing and orbit estimation approach, assuming that we are given prior densities $p_i(x_i)$ of states X_i and prior probabilities $\pi_i = \Pr(\mathsf{H}_i)$ of hypotheses H_i, $i = 0, 1$. Write $\delta(z)$ for a Dirac delta-function. Introduce the loss function

$$L_{j,i}(z_j, x_i) = \begin{cases} c & \text{if } i \neq j \\ c[1 - \delta(x_i - z_i)] & \text{if } i = j \end{cases} \tag{27}$$

which characterizes losses due to the decision $D_j = (j, z_j)$ that the hypothesis H_j is true and the estimate z_j is used for the state X_j ($j = 0, 1$) when the hypothesis H_i is correct and x_i is a correct value of the state X_i ($i = 0, 1$). The value of $c > 0$ is a loss due to a wrong decision and inaccurate estimation.

For $i = 0, 1$, define the posterior densities of the states X_i given $Y = y$

$$p_{X_i|Y}(x_i|\mathsf{H}_i, y) = \frac{p_{Y|X_i}(y|\mathsf{H}_i, x_i)p_i(x_i)}{\int p_{Y|X_i}(y|\mathsf{H}_i, x_i)p_i(x_i)\mathrm{d}x_i}.$$

Using the standard decision-theoretic approach it can be shown [13] that, under the loss function (27), the optimal Bayesian joint track identification-estimation-fusion rule D has the form:

$$D = \begin{cases} D_1 = (1, x_1^*) & \text{if } \Lambda(y) \geqslant \frac{1-\pi_1}{\pi_1} \frac{p_{X_0|Y}(x_0^*|\mathsf{H}_0, y)}{p_{X_1|Y}(x_1^*|\mathsf{H}_1, y)} \\ D_0 = (0, x_0^*) & \text{otherwise} \end{cases}, \tag{28}$$

where

$$\Lambda(y) = \frac{\int p_{Y|X_1}(y|\mathsf{H}_1, x_1)p_1(x_1)\mathrm{d}x_1}{\int p_{Y|X_0}(y|\mathsf{H}_0, x_0)p_0(x_0)\mathrm{d}x_0} \tag{29}$$

is the likelihood ratio and

$$x_i^* = \arg\max_{x_i} p_{X_i|Y}(x_i|\mathsf{H}_i, y), \quad i = 0, 1 \tag{30}$$

are maximum posterior estimates (see [13] for the details).

Note that the optimal identification-estimation rule (28)–(30) solves the problem of hypothesis testing and orbit estimation jointly and allows for the fusion of two tracks of the same object observed in different sessions in an optimal way. In general, the threshold in this rule is random (depends on the accuracy of estimation).

Under the linear approximation relative to the point $\hat{x}_i$, where $\hat{x}_0$ is calculated by the method described in Section 2.2 and components $\hat{x}_{1,0}, \hat{x}_{1,1}$ of $\hat{x}_1 = (\hat{x}_{1,0}^T, \hat{x}_{1,1}^T)^T$ are calculated as in Section 2.1, we have

$$H_i(x_i) \approx b_i + B_i x_i, \tag{31}$$

where

$$B_i = \frac{\partial H_i}{\partial x_i}(\hat{x}_i).$$

Under this approximation the likelihood functions $p_{Y|X_i}(y|\mathsf{H}_i, x_i), i = 0, 1$ are Gaussian

$$p_{Y|X_i}(y|\mathsf{H}_i, x_i) = \varphi_{b_i + B_i x_i, \widehat{W}}(y),$$

where

$$\varphi_{\mu,W}(x) = \frac{1}{\sqrt{(2\pi)^k |W|}} \exp\left\{-\frac{1}{2}(x-\mu)^T W^{-1}(x-\mu)\right\}$$

denotes density of the k-dimensional normal distribution with the mean μ and covariance matrix W.

For $i = 0, 1$, introduce the following notation

$$\mu_i = (B_i^T \widehat{W}^{-1} B_i)^{-1} B_i^T \widehat{W}^{-1}(y - b_i),$$

$$F_i = (B_i^T \widehat{W}^{-1} B_i)^{-1},$$

$$R_i(y) = (y - b_i - B_i\mu_i)^T \widehat{W}^{-1}(y - b_i - B_i\mu_i), \tag{32}$$

$$\Delta(y) = R_0(y) - R_1(y). \tag{33}$$

Assume that the prior densities $p_i(x) = I_{S_i}(x)/V_i$ of the states $X_i, i = 0, 1$ are uniform on hyperballs S_i with volumes $V_i = \delta\hat{V}_i$, where $\hat{V}_i$ are some positive finite numbers and $\delta > 0$. Hereafter $I_S(x)$ stands for the indicator function of the set S, i.e., $I_S(x) = 1$ if $x \in S$ and 0 otherwise. It can be shown [13] the optimal identification-estimation rule under the aforementioned linearization and as $\delta \to \infty$, that is, when the priors become improper uniform on hyperballs with infinite volumes is of the form:

$$D = \begin{cases} (1, \mu_1) & \text{if } \Delta(y) \geqslant T \\ (0, \mu_0) & \text{otherwise} \end{cases}, \tag{34}$$

where $\Delta(y)$ is defined in (33) and

$$T = 2\log\left(\frac{\hat{V}_1}{\hat{V}_0} \frac{1 - \pi_1}{\pi_1}\right). \tag{35}$$

Therefore, the hypothesis H_1 is accepted when the distance $\Delta(y)$ between the objective functions becomes relatively large.

It turns out that threshold T does not depend on the data. For practical purposes, it is better to determine this constant threshold based on the given probability of error. To this end, note that the statistic $\Delta(Y)$ can be represented in the form

$$\Delta(Y) = \varepsilon^T \widehat{W}^{-1}[B_0(B_0^T \widehat{W}^{-1} B_0)^{-1} B_0^T - B_1(B_1^T \widehat{W}^{-1} B_1)^{-1} B_1^T]\widehat{W}^{-1}\varepsilon.$$

It can be shown that the statistic $\Delta(Y)$ under the hypothesis H_0 is distributed according to the chi-squared distribution χ_k^2 with k degrees of freedom, where k is the dimensionality of the vector X_1. Therefore, the probability of misidentification $\mathsf{P}(\text{accept } \mathsf{H}_1|\mathsf{H}_0)$ is

$$P_I = \mathsf{P}\{\Delta(Y) > T|\mathsf{H}_0\} = 1 - \frac{\gamma(k/2, T/2)}{\Gamma(k/2)},$$

where $\Gamma(n)$ is a Gamma-function with n degrees of freedom and $\gamma(n,t)$ is the lower incomplete gamma function. This relation allows us to find the probability of false acceptance of the hypothesis H_1 as a function of threshold, $P_I(T)$, which in turn allows for finding the threshold to guarantee the required probability of false identification.

An alternative approach usually used in practice is solving hypothesis testing and estimation problems separately and applying the generalized likelihood ratio test, which is based on thresholding the generalized likelihood ratio statistic [16]:

$$\frac{\sup_{x_1} p_{Y|X_1}(y|\mathsf{H}_1, x_1)}{\sup_{x_0} p_{Y|X_0}(y|\mathsf{H}_0, x_0)} \gtrless C,$$

where threshold C does not depend on the accuracy of estimation. The latter popular test is not optimal.

The experiments with real data and simulations show that the developed identification-estimation algorithm guarantees reliable identification and orbit estimation even for highly separated tracks. Table 5 shows one of the results of estimation of orbit parameters based on two tracks separated by 24 hours for the same objects with NORAD catalog numbers $25544, 42916, 43246$, for which Tables 2–4 in Section 2.1 illustrate accuracy based on a single track. Comparing the data in Table 5 with that in Tables 2–4 allows us to conclude that the root-mean-squared errors of estimation of both position and velocity significantly decreased after fusion of two tracks.

Table 5. Accuracy of estimation for objects with Norad Numbers 25544, 42916, 43246 based on two tracks

Object	T (sec)	σ_{pos} (km)	σ_{vel} (m/sec)
25544	120	$0,003453$	$0,046$
42916	600	$0,061263$	$0,039$
43246	1200	$0,045443$	$0,017$

4. ADDITIONAL EXPERIMENTS

In addition, we performed an extensive analysis for a number of space objects moving in different orbits such as highly elliptical orbit (HEO) satellites without noticeable deceleration in the atmosphere and with deceleration in the atmosphere, as well as low-Earth orbit (LEO) satellites without a noticeable deceleration in the atmosphere and with a noticeable deceleration in the atmosphere.

Here we present results only for one but challenging case of a HEO space object with an apogee of $19,000$ km and a perigee of 360 km, for which deceleration in the atmosphere is noticeable. This object is not in the NORAD database. For the analysis, three tracks of this object were selected, obtained over a 30-day period with telescopes of the UN ORT network, located in Blagoveshchensk and Ussuriysk (Russia). Three tracks were observed: Track 1 (November 07, 2018, Blagoveshchensk), observation range $14,000$ km, observation duration 52 sec; Track 2 (November 20, 2018, Blagoveshchensk), observation range $14,000$ km, observation duration 80 sec; Track 3 (December 7, 2018, Ussuriysk), observation range $19,000$ km, observation duration 125 sec.

The quality of the identification-estimation algorithm was evaluated by the average of the angular discrepancy normalized to the total number of measurements for all tracks (empirical standard error of measuring the angular coordinates):

$$\hat{\sigma}(x) = \sqrt{\frac{[y - h(x)]^T [y - h(x)]}{\sum_{i=1}^K n_i - k}}, \tag{36}$$

where n_i is the number of measurements for the ith track and K is the number of fused tracks. Also, for each angular measurement, we determined the values of the longitudinal and transverse (relative to the direction of the velocity vector) deviation (in km) of the estimate of the position of the object from the bearing beam (direction from the telescope to the object determined by the measured angles).

Two experiments were performed. In the first experiment, to build the orbit we fused two short tracks – Track 1 (52 seconds) and Track 2 (80 seconds), obtained by a telescope in Blagoveshchensk with an interval of 13 days. The empirical standard angular error (defined in (36)) was 1 arc-second, which is the best one can do since it is comparable with the telescope measurement angular errors. The longitudinal and transverse deviations of the orbit from each measurement obtained over the interval of 40 days are shown in Figure 2. Since it is impossible to estimate the area-to-mass ratio for two separated tracks, the error outside the estimation interval increases quadratically due to atmospheric deceleration and reaches a value of 220 km by the end of the three-week test interval. In the second experiment, Track 3 of duration of 125 seconds, obtained on December 7 with a telescope in Ussuriysk, was added to the previous two tracks. The empirical standard angular error (36) was 1 arc-second. Longitudinal and transverse deviations are depicted in Figure 3. It can be seen from the figure that the area-to-mass ratio was estimated satisfactorily and there is no quadratic increase in the prediction error. The deceleration in the atmosphere is predicted 2 weeks ahead relative to the end of the orbit estimation interval with an error less than 30 km along the orbit, that is, much more accurately than without estimating the area-to-mass ratio obtained from two tracks.

Therefore, the results of experiments allow us to conclude

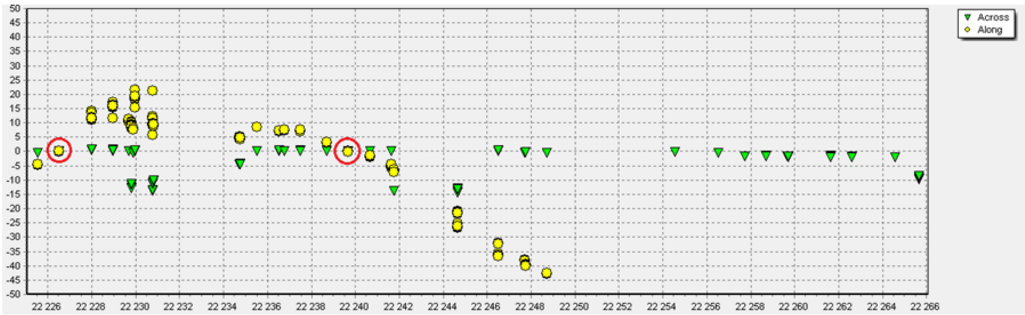

Figure 2. Longitudinal and transverse deviations of the orbit from measurements when fusing two tracks.

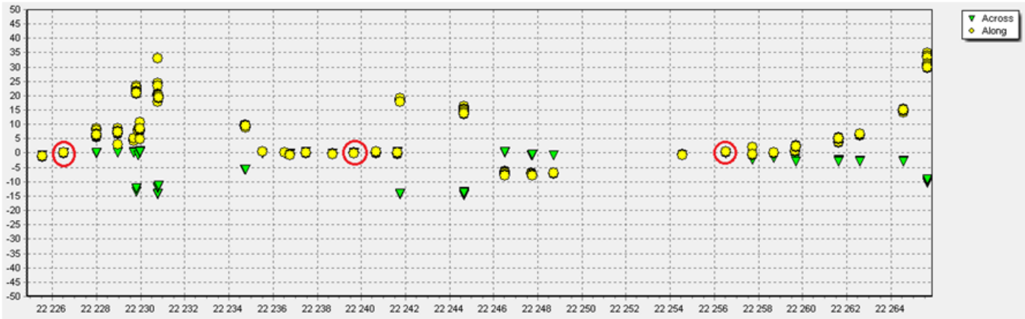

Figure 3. Longitudinal and transverse deviations of the orbit from measurements when fusing three tracks.

that even for tracks as short as $40 - 60$ sec separated by very long intervals ($2-3$ weeks) the proposed estimation algorithm consistently finds the global minimum of the objective function having several dozens of local minima.

5. CONCLUSION

We have described in detail a novel approach recently developed by Kolessa et al. [13] for identification of short time-separated optical tracks of an unknown near-Earth space object as well as for fusion of these tracks. This approach guarantees a nearly optimal (under assumed conditions) joint identification-fusion-estimation. The algorithm not only identifies the tracks as belonging to a single unknown object in an optimal manner (with a minimal probability of erroneous identification), but also estimates the parameters of the object orbit extremely accurately. At the same time, the algorithm is computationally simple. Fusion of two tracks on a regular laptop typically does not exceed 0.5 seconds.

The experimental results based on real data from UN ORT allow us to conclude that the accuracy of the orbit estimation based on the developed algorithms is extremely high and close to potentially achievable, even for very short tracks separated by 2 weeks and more. The experiments also show that even with very long pauses between observation sessions (up to $2-3$ weeks) and tracks obtained on a short orbit arc ($40 - 60$ sec), the proposed estimation algorithm is robust and consistently finds the global minimum of the objective function that has up to several dozens of local minima.

ACKNOWLEDGMENTS

The work was supported in part by the 5-100 excellence project and the Arctic program (Russian Ministry of Education and Science) as well as by the grant 18-19-00452 from the Russian Science Foundation at the Moscow Institute of Physics and Technology.

REFERENCES

1. Albert, A. (1972). *Regression and the Moor–Penrose Pseudoinverse*, ser. Mathematics in science and engineering, vol. 94. Academic Press, New York, USA.
2. Brent, R. (1973). *Algorithms for Minimization without Derivatives*, ser. Mathematics in science and engineering, vol. 94. Prentice-Hall, Englewood Cliffs, NJ, USA.
3. Bar-Shalom, Y., Li, X. R., and Kirubarajan, T. (2001). *Estimation with Applications to Tracking and Navigation: Theory, Algorithms and Software*. Wiley, New York, USA.
4. Chang, C. (1980). Ballistic trajectory estimation with angle-only measurements, *IEEE Transactions on Automatic Control*, **25**(3), 474–480.

5. Chen, H., Kirubarajan, T., and Bar-Shalom, Y. (2003). Performance limits of track-to-track fusion vs. centralized estimation: Theory and application, *IEEE Transactions on Aerospace and Electronic Systems*, **39**(2), 386–400.

6. DeMars, K. J., Jah, M. K., and Schumacher, P. W. (2012). Initial orbit determination using short-arc angle and angle rate data, *IEEE Transactions on Aerospace and Electronic Systems*, **48**(3), 2628–2636.

7. Ferguson, T. S. (1967). *Mathematical Statistics: A Decision Theoretic Approach*, ser. Probability and Mathematical Statistics. Academic Press, New York, USA.

8. Fujimoto, K. and Scheeres, D. J. (2010). Correlation of optical observations of earth-orbiting objects and initial orbit determination, *Proceedings of the 2010 AIAA/AAS Astrodynamics Specialist Conference (AIAA-2010-7975)*, Darmstadt, Germany.

9. Fujimoto, K. and Scheeres, D. J. (2011). Short-arc correlation and initial orbit determination for space-based observations, *Proceedings of the 2011 Advanced Maui Optical and Space Surveillance Technologies Conference (AMOS 2011)*, Maui, HW, USA.

10. Fujimoto, K. and Scheeres, D. J. (2013). Short-arc correlation and initial orbit determination for space-based observations, *Proceedings of the Sixth European Conference on Space Debris, Toronto, Ontario, Canada*.

11. Gauss, C. F. (1857). *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections: A translation of Gauss's "Theoria motus," with an appendix*, Little, Brown and Company, Boston, USA.

12. Kolessa, A.E., Ivanov, A.P., and Radchenko, V.A. (2014). Searching of unknown earth-orbiting object in the next observation session, *Proceedings of the International Conference on Engineering &Telecommunication (En&T), Moscow Institute of Physics and Technology, Moscow, Russia*, Published by the IEEE Computer Society.

13. Kolessa, A.E., Tartakovsky, A.G., Ivanov, A.P., Radchenko, V.A. (2019). Nonlinear estimation and decision-making methods in short track identification and orbit determination problem, *IEEE Transactions on Aerospace and Electronic Systems*, 2019 (submitted, under review).

14. Legendre, A.M. (1805). *Nouvelles methodes pour la determination des orbites des cometes*. F. Didot, Paris, France.

15. Milani, A., Gronchi, G., Vitturi, M., and Knezevic, Z. (2004). Orbit determination with very short arcs. Admissible regions, *Celestial Mechanics and Dynamical Astronomy*, **90**(1), 57–85.

16. Tartakovsky, A., Nikiforov, I., and Basseville, M. (2014). *Sequential Analysis: Hypothesis Testing and Changepoint Detection*, ser. Monographs on Statistics and Applied Probability. Chapman & Hall/CRC Press, Boca Raton, London, New York.

17. Tartakovsky, G.P. (2005). *The Theory of Information Systems*. Fizmatkniga, Moscow, Russia.

18. Tommei, G., Milani, A., and Rossi, A. (2007). Orbit determination of space debris: Admissible regions, *Celestial Mechanics and Dynamical Astronomy*, **97**(4), 289–304.